Zero-temperature renormalization method

Ising model in a transverse field in 1-d

$$\mathcal{H} = -\left(J\sum_i S_i^x S_{i+1}^x + h\sum_i S_i^z\right)$$

$$S^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad S^x = \frac{S^+ + S^-}{2}$$

states:

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv |\uparrow\rangle$ $\quad$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv |\downarrow\rangle$

$|+\rangle$ $\quad$ $|-\rangle$

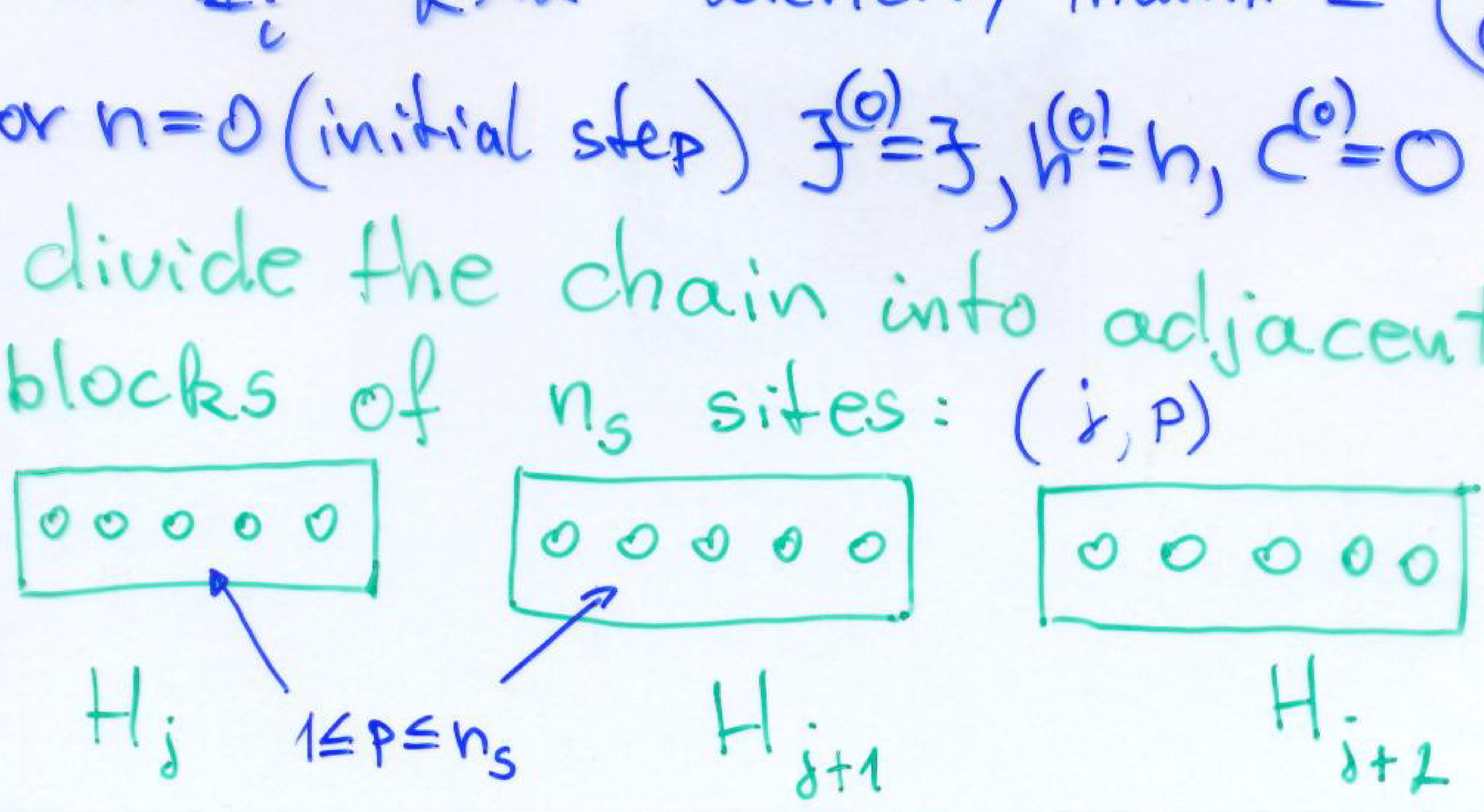
iterativ method:

$$H^{(n)} = -\sum_i J^{(n)} S_i^{x(n)} S_{i+1}^{x(n)} + h^{(n)} S_i^{z(n)} + C\sum_i I_i^{(n)}$$

where $I_i^{(n)}$ 2x2 identity matrix $\equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

for $n=0$ (initial step) $J^{(0)} = J$, $h^{(0)} = h$, $C^{(0)} = 0$

1) divide the chain into adjacent blocks of n_s sites: (j, p)

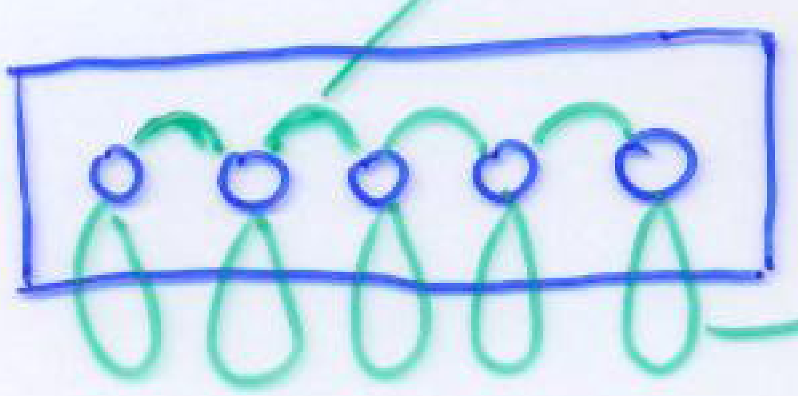


$S^{x(n)}_{j,p}$ $\quad i \equiv (j,p)$

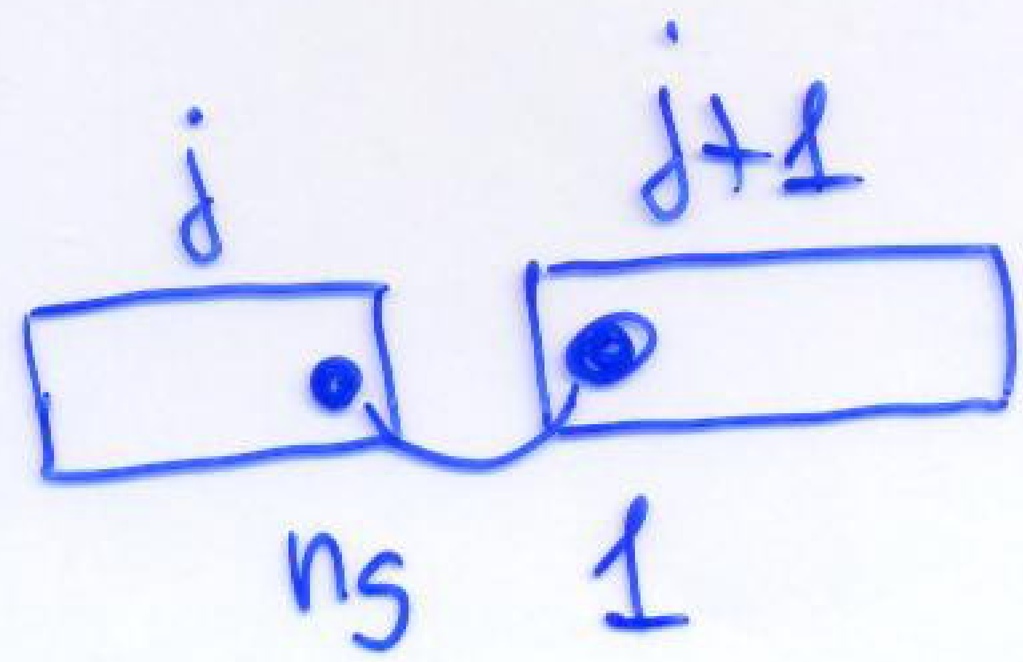
$$\mathcal{H}^{(n)} = \sum_j \left\{ \mathcal{H}^{(n)}_j + \mathcal{H}^{(n)}_{j,j+1} + C^{(n)} \sum_{p=1\dots n_s} I^{(n)}_{j,p} \right\}$$

Intrablock $\mathcal{H}$ $\qquad$ Interblock $\mathcal{H}$

$$\mathcal{H}^{(n)}_j = \underbrace{-J^{(n)} \sum_{p=1\dots n_s-1} S^{x(n)}_{j,p} S^{x(n)}_{j,p+1}} \underbrace{- h^{(n)} \sum_{p=1\dots n_s} S^{z(n)}_{j,p}}$$



$$\mathcal{H}^{(n)}_{j,j+1} = -J^{(n)} S^{x(n)}_{j,n_s} S^{x(n)}_{j+1,1}$$



First, we solve $\mathcal{H}^{(n)}_j$ exactly in space ϵ_j of dimensionality 2^{n_s} generated by basis vectors $|\epsilon_1, \epsilon_2, \epsilon_3, \dots \epsilon_{n_s}\rangle$ where $\epsilon_p \equiv +1, -1$ corresponding to the eigenstates of $S^{z(n)}_{j,p}$ in the block.

$\mathcal{H}_j$ acting on a basis vector does not change the parity of the total number of (+) or (−) signs.

$$\epsilon_j \begin{cases} \nearrow \epsilon_j^+ & \text{even \# of } (-) \\ \searrow \epsilon_j^- & \text{odd \# of } (-) \end{cases} \qquad 2^{n_s-1}$$

2) After diagonalization of $\mathcal{H}_j^{(n)}$ retain only the lowest energy states of $\epsilon_j^{+/-}$: $|+\rangle^{(n+1)}$ with $E_+^{(n+1)}$ and
$|-\rangle^{(n+1)}$ with $E_-^{(n+1)}$

$$|+\rangle^{(n+1)} = \sum{}^{+} \lambda^{+(n)}_{\epsilon_1 \dots \epsilon_p \dots \epsilon_{n_s}} |\epsilon_1 \dots \epsilon_p \dots \epsilon_{n_s}\rangle$$

$$|-\rangle^{n+1} = \sum{}^{-} \lambda^{-(n)}_{\epsilon_1 \dots \epsilon_p \dots \epsilon_{n_s}} |\epsilon_1, \dots \epsilon_p \dots \epsilon_{n_s}\rangle$$

$\sum^+$ and $\sum^-$ summations restricted to subspaces ϵ_j^+, ϵ_j^-

$\lambda^{\pm(n)}_{\epsilon_1 \dots \epsilon_{n_s}}$, $E_\pm^{(n+1)}$ determined by machine

$E_+^{(n+1)} < E_-^{(n+1)}$ and always the lowest levels of the whole spectrum of $\mathcal{H}_j^{(n)}$

3) New operators $S_j^{x(n+1)}$, $S_j^{z(n+1)}$

$$\mathcal{H}_j^{(n)} = -h^{(n+1)} S_j^{z(n+1)} + \frac{1}{2}\left(E_+^{(n+1)} + E_-^{(n+1)}\right) I_j^{(n+1)}$$

with

$$h^{n+1} = \frac{1}{2}\left(E_-^{(n+1)} - E_+^{(n+1)}\right)$$

Taking matrix elements of old spin $S_{j,p}^{x(n)}$ between $|+\rangle^{(n+1)}$ and $|-\rangle^{n+1}$ $\Rightarrow$ spin recursion relation:

$$S_{j,p}^{x(n)} = \xi_p^{(x)(n)} S_j^{x(n+1)}$$

where $\xi_p^{x(n)} = \sum^+ \lambda^{+(n)}_{\epsilon_1 \dots \epsilon_p \dots \epsilon_{n_s}} \lambda^{-(n)}_{\epsilon_1 \dots -\epsilon_p \dots \epsilon_{n_s}}$

$\xi_p^{x(n)}$, $h^{(n)}$, $J^{(n)}$ can be evaluated at each iteration for symmetry reasons:

$$\xi^{x(n)}_{n_s-p+1} = \xi^{x(n)}_{p} \quad \text{and especially} \quad \xi^{x(n)}_{n_s} = \xi^{x(n)}_{1}$$

$\Rightarrow$ Interblock Hamiltonian:

$$\mathcal{H}^{(n)}_{j,j+1} = -J^{(n+1)} S^{x(n+1)}_{j} S^{x(n+1)}_{j+1} \quad \text{where}$$

$$J^{(n+1)} = \left(\xi^{x(n)}_{1}\right)^2 J^{(n)}$$

$$C^{(n+1)} = n_s C^{(n)} + \frac{1}{2}\left(E^{(n+1)}_{+} + E^{(n+1)}_{-}\right)$$

- recursion relations + initial condidions define a renormalization group transformation.

$\left(J_0 = J,\ h^{(0)} = h,\ C^0 = 0\right)$

$h^{(n)} \approx$ splitting of the two lowest levels of the system and

E_0/N of the ground state as $N \to \infty$

$$\left(E_0/N\right)_{N\to\infty} = \lim_{n\to\infty}\left(C^{(n)}/n_s^{(n)}\right)$$

Calculations for different $n_s = 2, 3, 4 \ldots$

$\Rightarrow$ general features of transition recovered with $n_s = 2^{(1)}$ already !

1) $h/J < (h/J)_{Critical}$

- for $n \to \infty$ $h^{(n)} \to 0$, $J^n \to J^\infty \neq 0$
- the ground state is dublet
- RG trajectory converges ~~conver~~ to a fixed point corresponding to the simple Ising chain, i.e., without h

2) $h/J > (h/J)_{Critical}$

- for $n \to \infty$ $h^{(n)} \to h^\infty \neq 0$

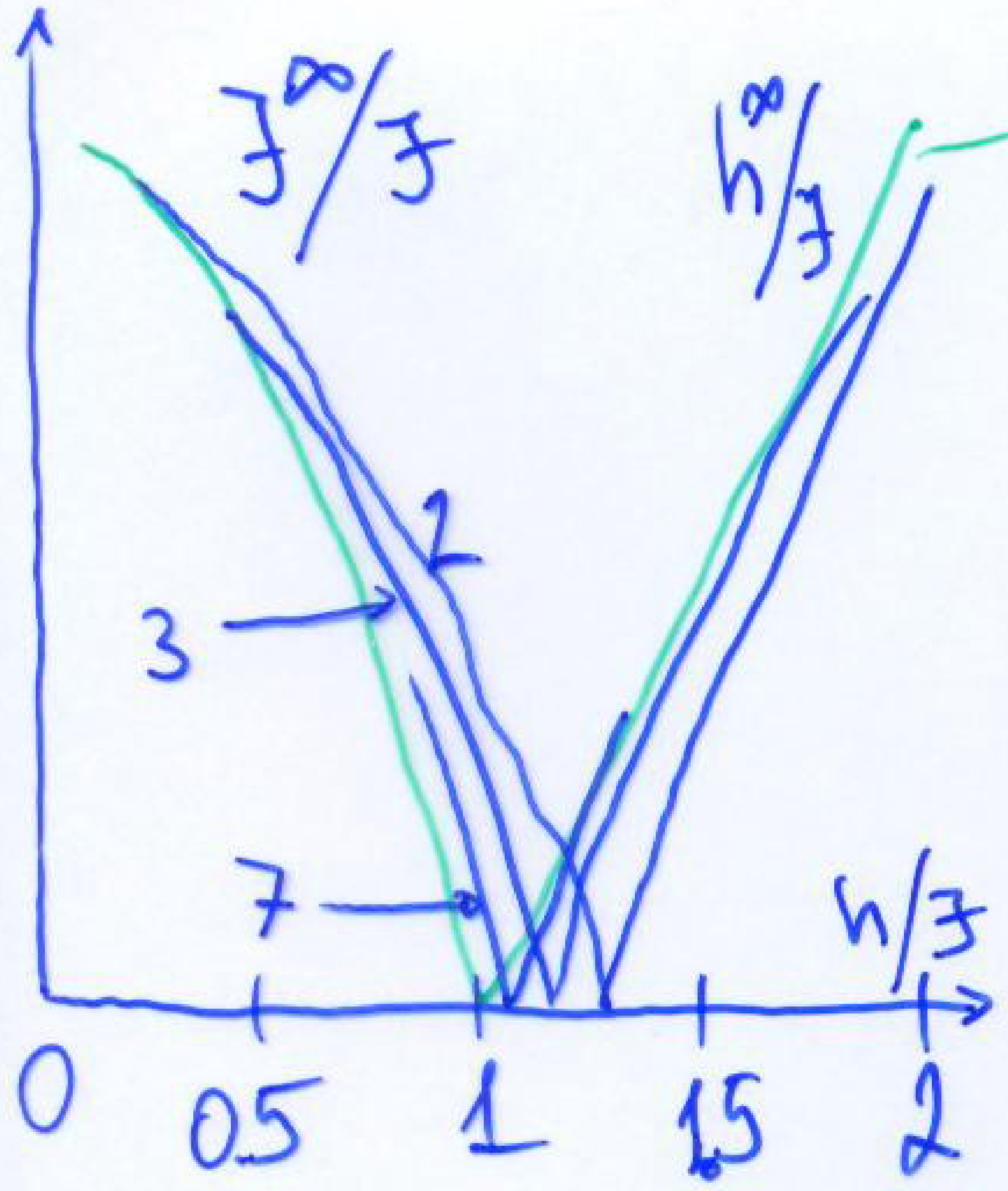
  $J^{(n)} \to 0$

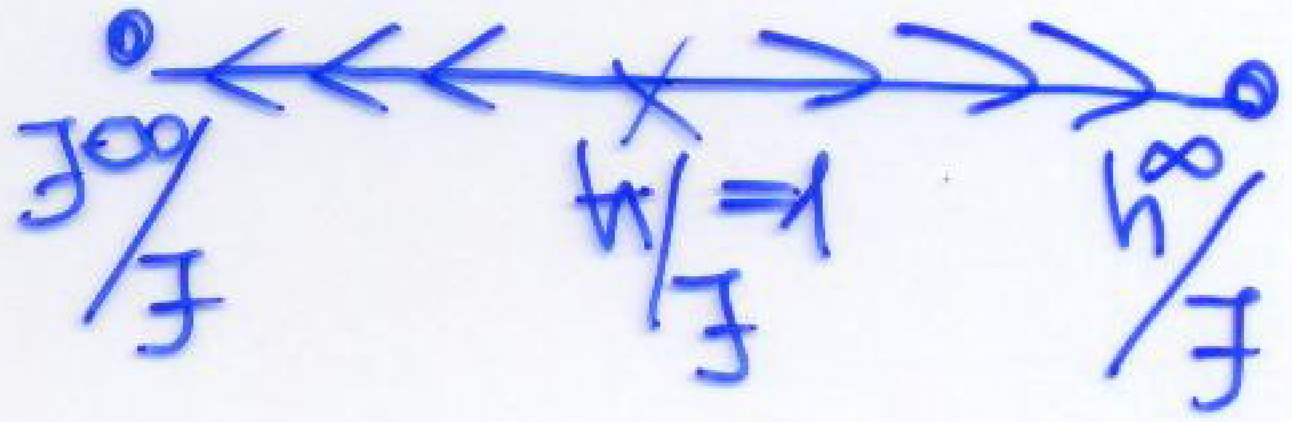
- ground state is singlet,
- energy gap $\Delta = 2h^\infty$
- RG-traj. $\Rightarrow$ chain of independent spins in h

- In one-parameter space h/J $(h/J)_c$ is an unstable fixed point separating two stable fixed points $h/J = 0$ and $h/J = \infty$

- In the two parameters space, h, J there are two fixed lines $h = 0$ and $J = 0$ corresponding to the two fixed point Hamiltonian.

- h^∞/J and J^∞/J are only functions of h/J





A) $h^\infty/J = \Delta/2J$ becomes more linear with increasing n_s

exact relation: $\Delta = 2(h-J)$

critical exponent

$\Delta \sim (h-h_c)^s$ $\quad s \xrightarrow[n_s\to\infty]{} 1$

n_s	s
2	1.276
⋮	
7	1.053
∞	1

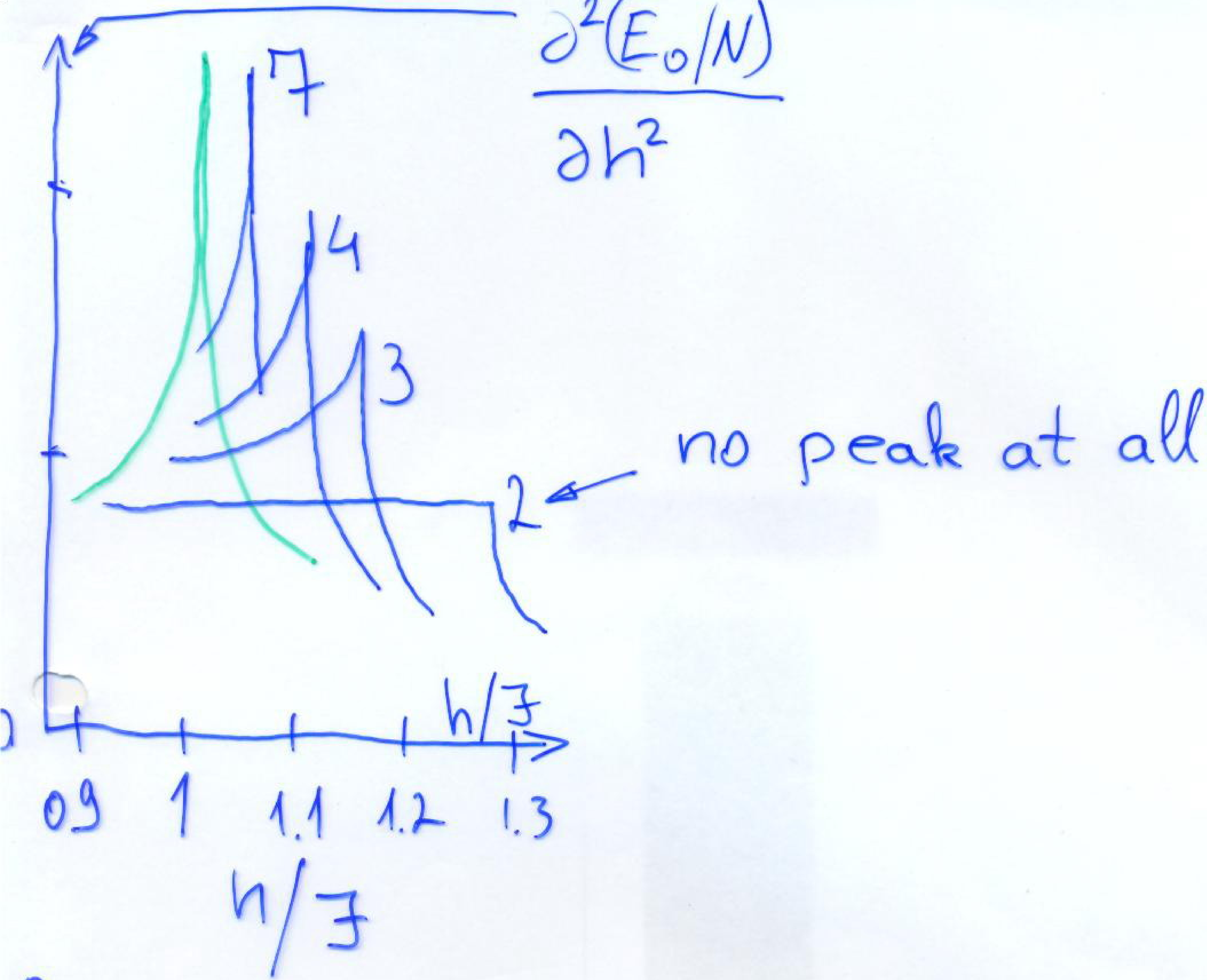
B) $J^\infty = J\left[1-(h/J)^2\right]$

parabolic → end to end x−x correlat.

C) E_0/N

z magnetic susceptibility

$$\frac{\partial\langle 0|S^z|0\rangle}{\partial h} = \frac{\partial^2 E_0/N}{\partial^2 h}$$



- logarithmic divergence at $h/J = 1$ develops with increasing n_s

Magnetization components

$\langle 0|S_i^x|0\rangle$, $\langle 0|S_i^y|0\rangle$, $\langle 0|S_i^z|0\rangle$; $|0\rangle \equiv GS$

$$\langle S_i^x\rangle = \xi_{P_0}^{x(0)}\, \xi_{P_1}^{x(1)}\, \xi_{P_2}^{x(2)} \cdots\, \xi_{P_{n-1}}^{x(n-n)} \langle S_j^{x(n)}\rangle$$

$P_0, P_1, \dots P_{n-1}$ depend on the location of $\vec{S}_i$ in the superblock of index j

$\langle S_i^y\rangle =$

$\langle S_i^z\rangle =$

for

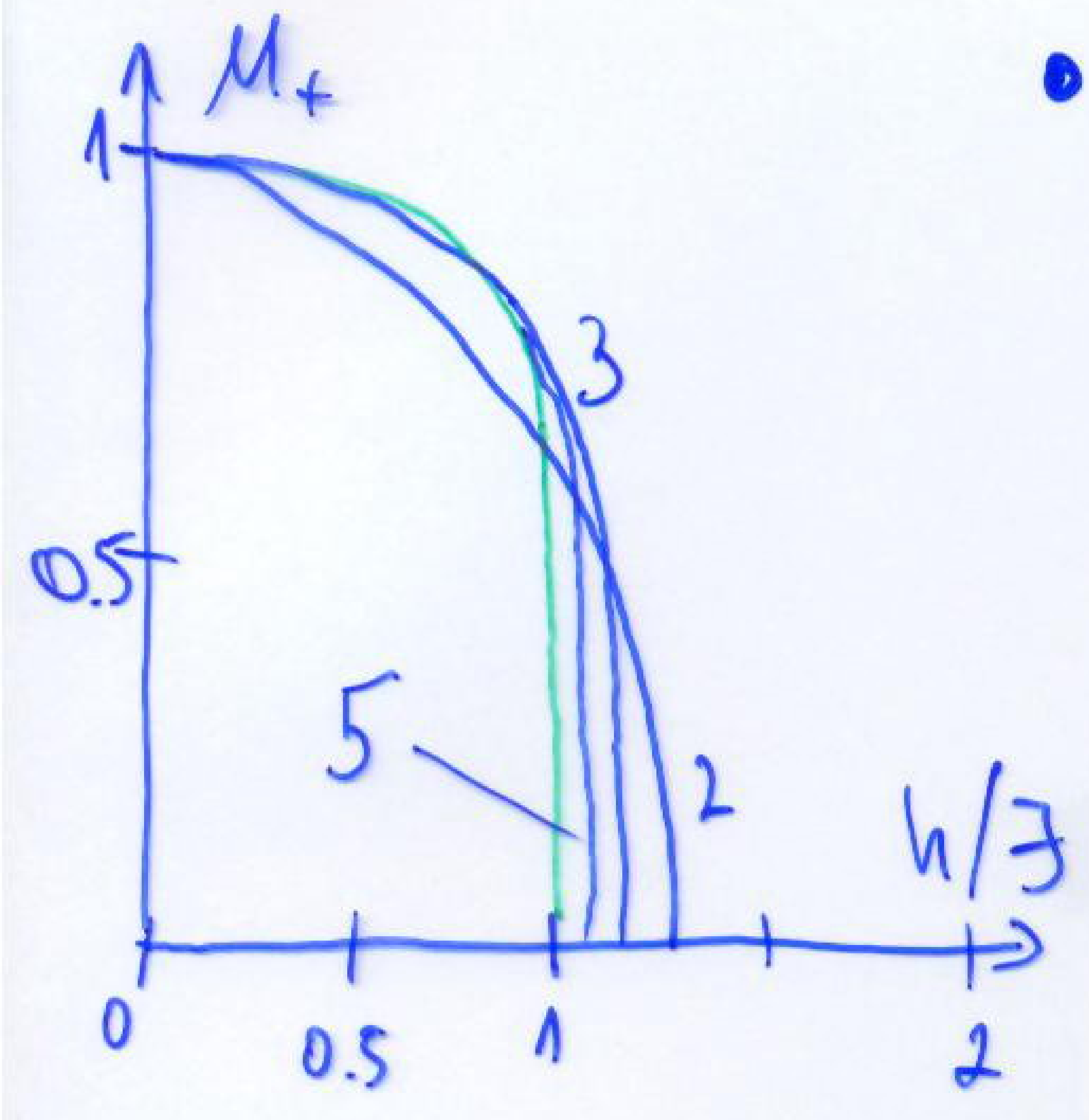
1) $h/J < (h/J)_c$

$\langle S^{x(n)}\rangle = \pm 1$ → no field so ± 1 can not be distinguished

$\langle S^{y(n)}\rangle = \langle S^{z(n)}\rangle = 0$

2) $h/J > (h/J)_c$ free spins in field

$\langle S^{x(n)}\rangle = \langle S^{y(n)}\rangle = 0$ but $\langle S^{z(n)}\rangle = 1$



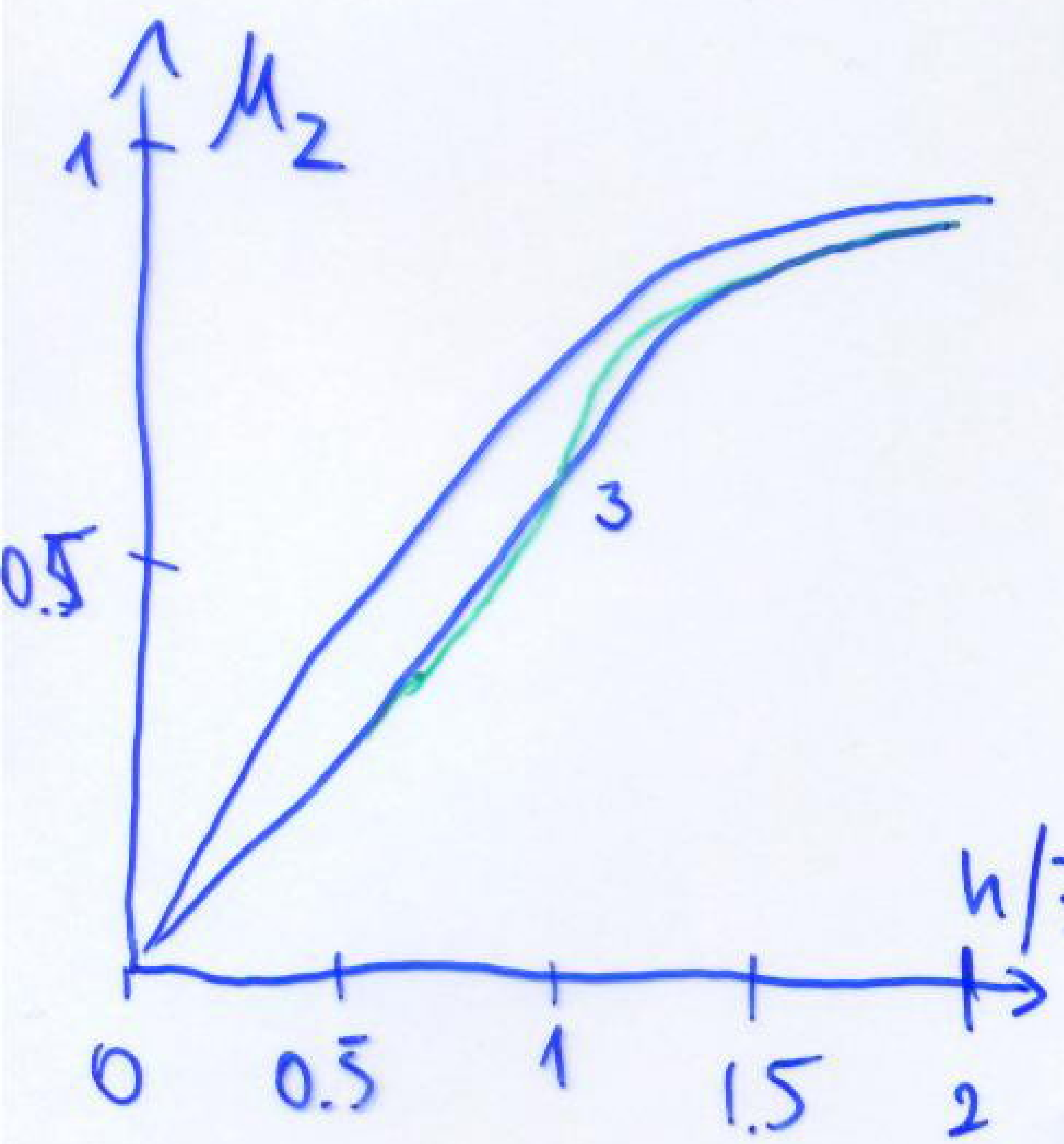
- $M_x = \left(\left(h/J\right)_c - h/J\right)^{\beta}$

$\beta_{exact} = 0.125$

for $n_s = 2$ $\quad \beta = 0.4$

$n_s = 7$ $\quad \beta = 0.145$

- $$\langle S_i^x \rangle = \begin{cases} \lim_{n \to \infty} \xi^{x(0)}_{P(0)} \xi^{x(1)}_{P(1)} \cdots \xi^{x(n)}_{P_n} & \\ 0 & h/J > \left(h/J\right)_c \end{cases}$$



- Spins at the edges of the blocks not correctly included

R. Jullien... PRB 18, 3568 (1978)

S.D. Drell... PRB 16, 1769 (1977)

R. Jullien... PRB 19 4646 (1979)

PRB 16 4889 (1977), PRB 19 4653 (1979)

Correlation functions

$\langle 0 | S_i^x S_{i'}^x | 0 \rangle$, $\langle | S^y S^y | \rangle$, $\langle | S^z S^z | \rangle$

$\begin{cases} \text{algebratic} \\ \text{exponential} \end{cases}$

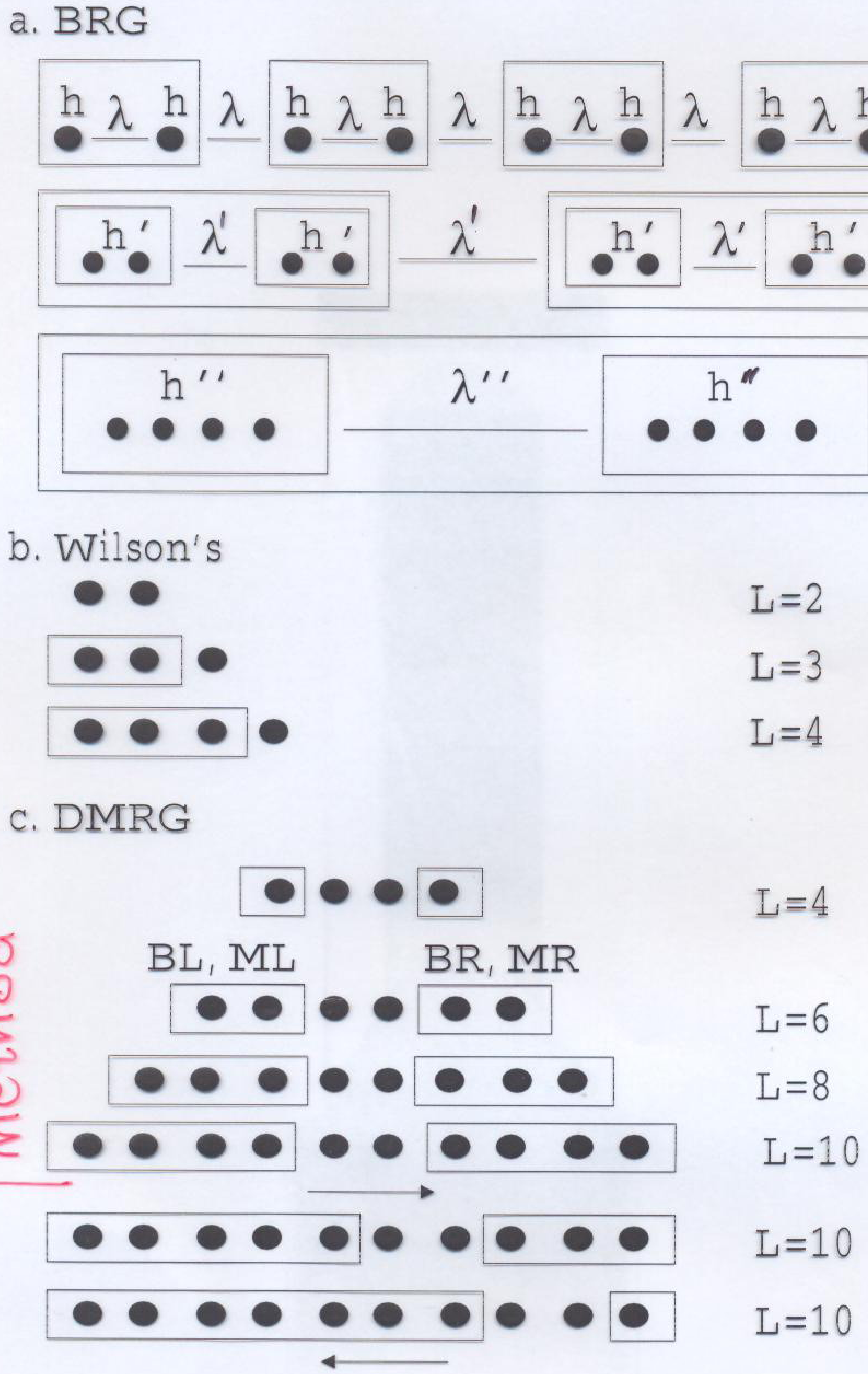
Correlation length diverges

$$R_c \sim \left[h/J - (h/J)_c \right]^{-\nu}$$

h^n/J^n can be linearized around the fixed point

$$\frac{h^{(n+1)}}{J^{(n+1)}} - \left(\frac{h}{J}\right)_c = n_s^y \left(\frac{h^{(n)}}{J^{(n)}} - \left(\frac{h}{J}\right)_c \right)$$

$\nu = 1/y \qquad \nu \to 1 \quad n_s \to \infty$



Infinite lattice method

Finite lattice m.

FIG. 1: Schematic plot of the spin couplings in the BRG, Wilson's and DMRG renormalization methods.