Classical data compression

- messages constructed from N indep. letters (x_α) with p_α a priory probability

- message $\to$ typical sequences / $\to$ atypical sequences

- if for $\epsilon > 0$ sum of probab. of all typ. sec. is between $1-\epsilon$ and 1 then for and $\delta > 0$ the number of typical sequences $n(\epsilon, \delta)$:

$$2^{N(S+\delta)} \geq n(\epsilon, \delta) \geq (1-\epsilon) 2^{N(S-\delta)}$$

for large enough N ans

$$S = -\sum_\alpha p_\alpha \log_2 p_\alpha .$$

where S is the Shannon entropy.

- Therefore a block code of NS bits encodes all typical sequences regardles how the atyp. sec.s are encoded the probability of error still $< \epsilon$.

DMRG

- Quantum Data Compression
- Relevant, irrelevant information
- Finite sysem (N), $\dim \Lambda = q^N$
- Selection rules (quantum numbers)

$q^l \to n$ pl.: $\binom{N}{N_\uparrow}\binom{N}{N_\downarrow} = n$

- Typical, non-typical subspaces:

$$\begin{array}{ccc} & \nearrow & \Lambda_{typ} \\ \Lambda & & \\ & \searrow & \Lambda_{nontyp} \end{array}$$

$$\mathrm{Tr}\, \Pi_{typ}\, \varsigma^N\, \Pi_{typ} > 1 - \epsilon$$

$$\mathrm{Tr}\, \Pi_{nont}\, \varsigma^N\, \Pi_{nont} < \epsilon$$

where $\varsigma^N = \varsigma \otimes \varsigma \otimes \dots \otimes \varsigma$

- message formed from N letter st.

$$\rho = \sum_\alpha w_\alpha |\varphi_\alpha\rangle\langle\varphi_\alpha|$$

ensemble $\{|\varphi_\alpha\rangle, w_\alpha\}$

$$\Lambda^N = \Lambda \otimes \Lambda \otimes \cdots \otimes \Lambda$$

- If ρ_i-s are independent : $S(\rho^N) = N S(\rho)$

$$S = \mathrm{Tr}\, \rho(\ln \rho) \quad \text{Neumann entropy}$$

$$\Lambda_{typ} : \quad \cancel{2}e^{-N(S(\rho)+\delta)} < w_\alpha < \cancel{2}e^{-N(S(\rho)-\delta)}$$

$$(1-\epsilon)\, \cancel{2}e^{-N(S(\rho)-\delta)} \leq \dim \Lambda_{Typ} \leq \cancel{2}e^{-N(S(\rho)+\delta)}$$

for arbitrary chosen small ϵ, δ

- Coding:

- letter states $|\varphi_\alpha\rangle$, w_α, $\rho = \sum_a w_\alpha |\varphi_\alpha\rangle\langle\varphi_\alpha|$
- $C(n, N)$ code words: $|\psi_i\rangle$, P_{ψ_i}

A code consists two things

- (i) a set of n code words $|\psi_i\rangle$: $i = 1 \ldots n$ where each $|\psi_i\rangle$ is a {sequence (i.e. product) of N letter states

(ii) not all product state ~~\$~~ used as code words → subset of states

a *priori* probability P_{ψ_i}

- The tolerance of a code is

$\tau = \max_{\varphi_\alpha} |f_{\varphi_i} - P_{\psi_i}|$ where

$f_{\psi_i} = \frac{1}{N} \sum_{i=1}^{n} P_{\psi_i} n_{\varphi_\alpha \psi_i}$ where

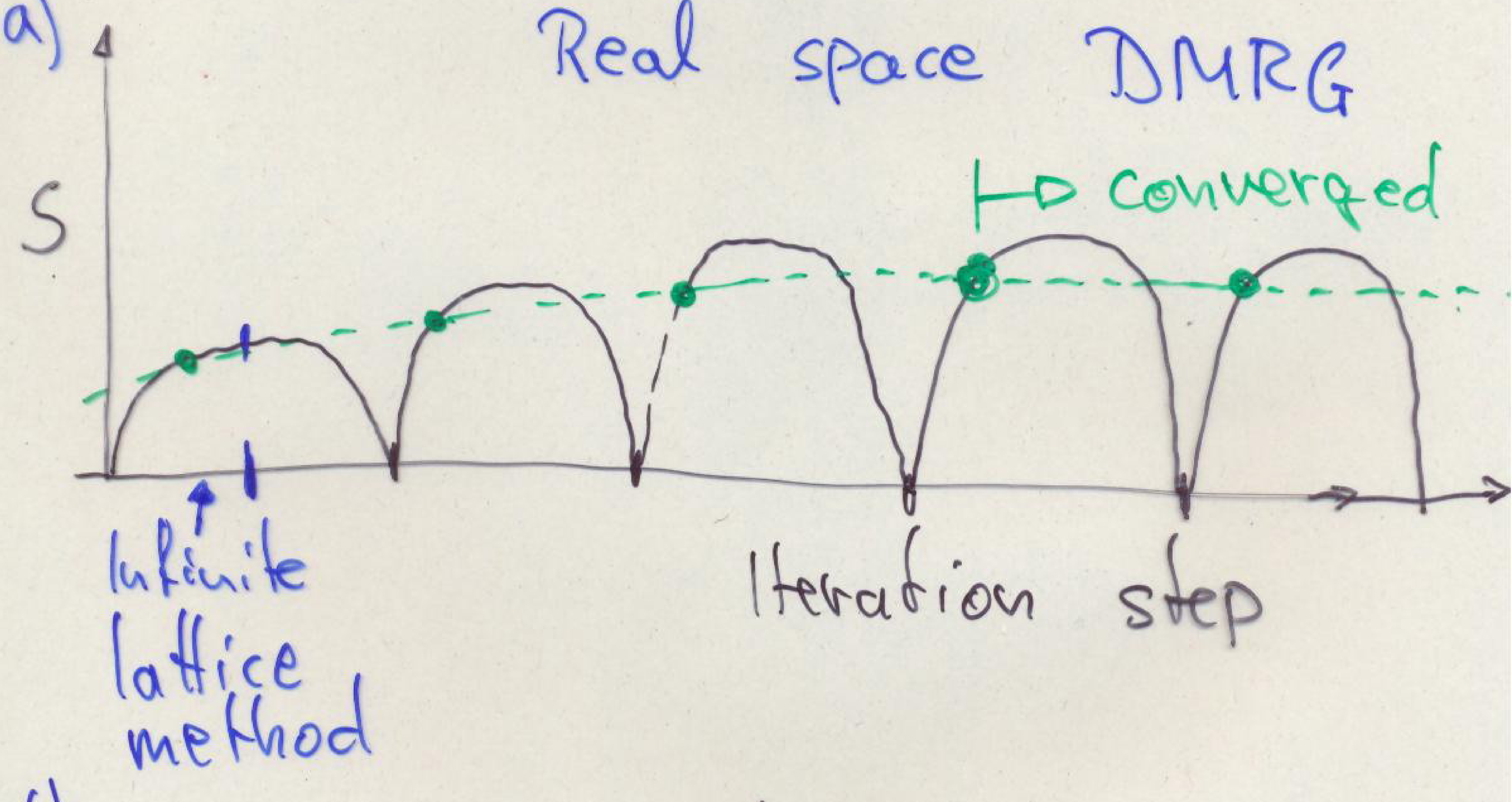
$n_{\varphi_\alpha \psi_i}$ is the number of occurence of letter $|\varphi_\alpha\rangle$ in code word $|\psi_i\rangle$

- Classical channel capacity is the maximum attainable mutual information where max is taken by all possible probability distribution of input letters
- for quantum channel if we allow w_α to vary then the channel capacity is $C = \max_{w_\alpha} S(\rho)$
  where S is the von Neumann en
- In DMRG w_α and φ_α and ψ_i are determined from the target state

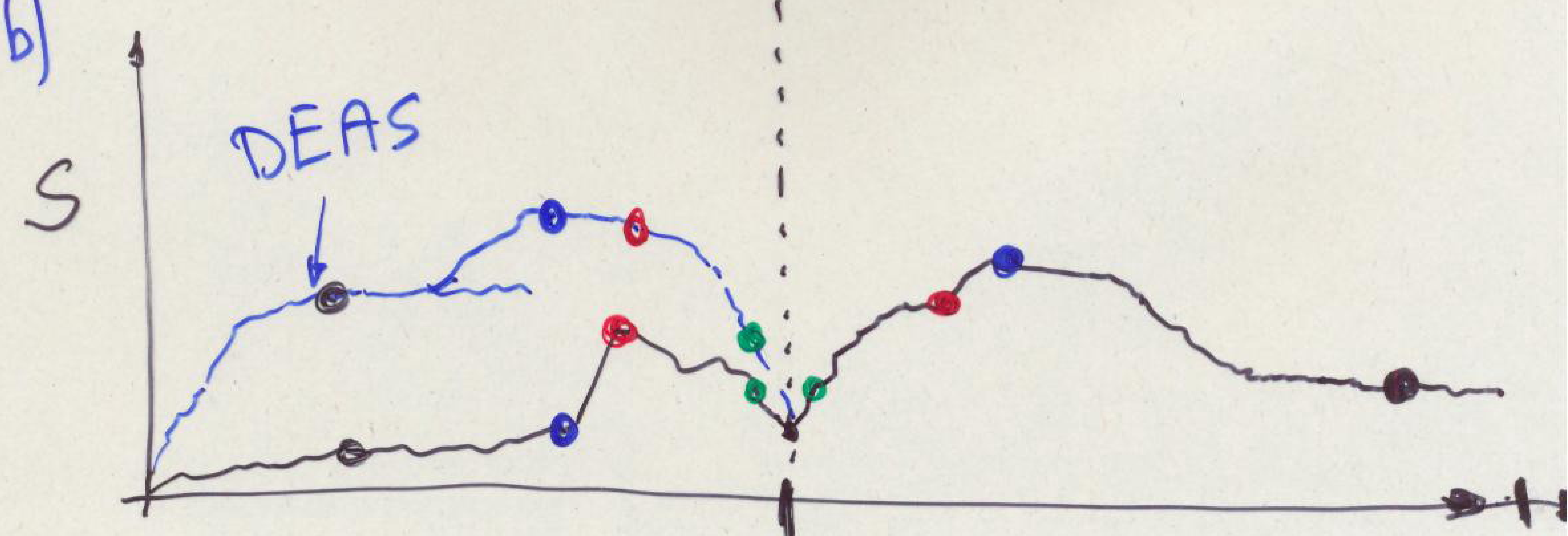
- code words change at each iteration step

- purpose of DMRG: it maximizes the Channel capacity for each iteration step.

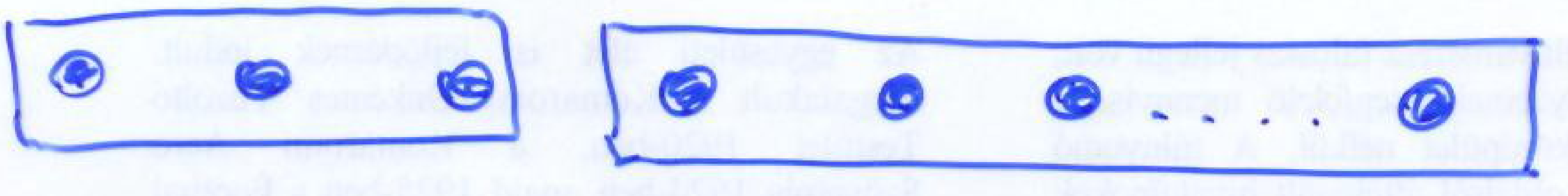
a)



b)



• DMRG

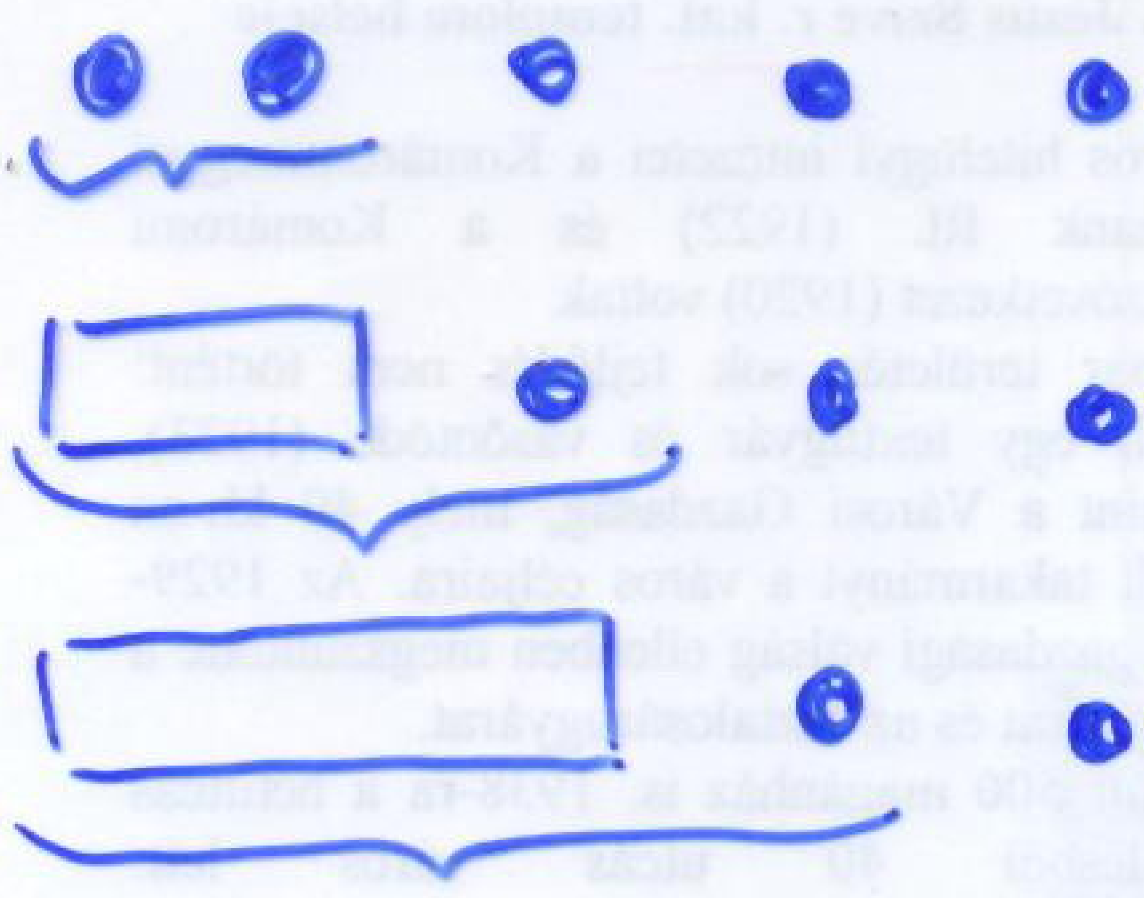


Λ_L, M_L $\qquad$ Λ_R, M_R

$$\Lambda = \Lambda_L \otimes \Lambda_R$$

$$\Psi_{TG} = \sum_{i=1}^{n} P_{\psi_i} |\psi_i\rangle, \quad \sum_i P^2_{\psi_i} = 1$$

$$q^N \xrightarrow{\text{Model}} n \xrightarrow{\text{DMRG}} \dim \Lambda_{TG} \Leftarrow \text{Typical subspace}$$



$$\rho_L = \mathrm{Tr}_R(\rho)$$

$$\rho_R = \mathrm{Tr}_L(\rho)$$

$$S_L = S_R = S$$

$\vdots$

$$A_{l+1} = O A_l O^+ \qquad O = \begin{pmatrix} |\psi^L_1\rangle \\ \vdots \\ |\psi^L_M\rangle \end{pmatrix}$$

$$\mathrm{Tr}E = 1 - \sum_{\alpha=1}^{M} w_\alpha$$

5

$$\mathcal{H} = t\left(c_i^+ c_{i+1} + c_{i+1}^+ c_i\right) + U c_i^+ c_i c_{i+1}^+ c_{i+1}$$

$q = 4$; $|\phi_\alpha\rangle \equiv |0\rangle\ |\downarrow\rangle\ |\uparrow\rangle\ |\downarrow\uparrow\rangle$

PBC, OBC

$U = 0 \quad S_i = 4 \ln 4 = 1.38..$

$U = \infty \quad S_i = 2 \ln 2 = 0.69... \quad |\uparrow\rangle, |\downarrow\rangle$

- Compressibility : $\gamma \equiv \dfrac{\dim \Lambda_{TG}}{n}$

- $\log(\dim \Lambda_{TG}) = \cancel{\beta} \cdot S = \beta$ (shift)

3

$|\Psi_g\rangle$. Thus, the aim in the following is to compute the entropy of entanglement, Eq. (1), for the state $|\Psi_g\rangle$ according to bipartite partitions parametrized by L,

$$S_L \equiv -\mathrm{tr}(\rho_L \log_2 \rho_L), \tag{4}$$

where $\rho_L \equiv \mathrm{tr}_{\mathcal{B}_L} |\Psi_g\rangle\langle\Psi_g|$ is the reduced density matrix for $\mathcal{B}_L$, a block of L spins. The motivation behind the present approach is straightforward: by considering the entanglement S_L of a spin block as a function of its size L, and by characterizing it for large L, one expects to capture the large-scale behavior of quantum correlations at a critical regime.

We start off with a description of the calculations, to then move to the analysis and discussion of the results, a summary of which is provided by Fig. 1. The XXZ model, Eq. (2), can be analyzed by using the Bethe ansatz [15]. We have numerically determined the ground state $|\Psi_g\rangle$ of H_{XXZ} for a chain of up to $N = 20$ spins, from which S_L can be computed. We recall that in the XXZ model, and due to level crossing, the nonanalyticity of the ground-state energy characterizing a phase transition already occurs for a finite chain. Correspondingly, already for a chain of $N = 20$ spins it is possible to observe a distinct, characteristic behavior of S_L depending on whether the values (Δ, λ) in Eq. (2) belong or not to a critical regime.

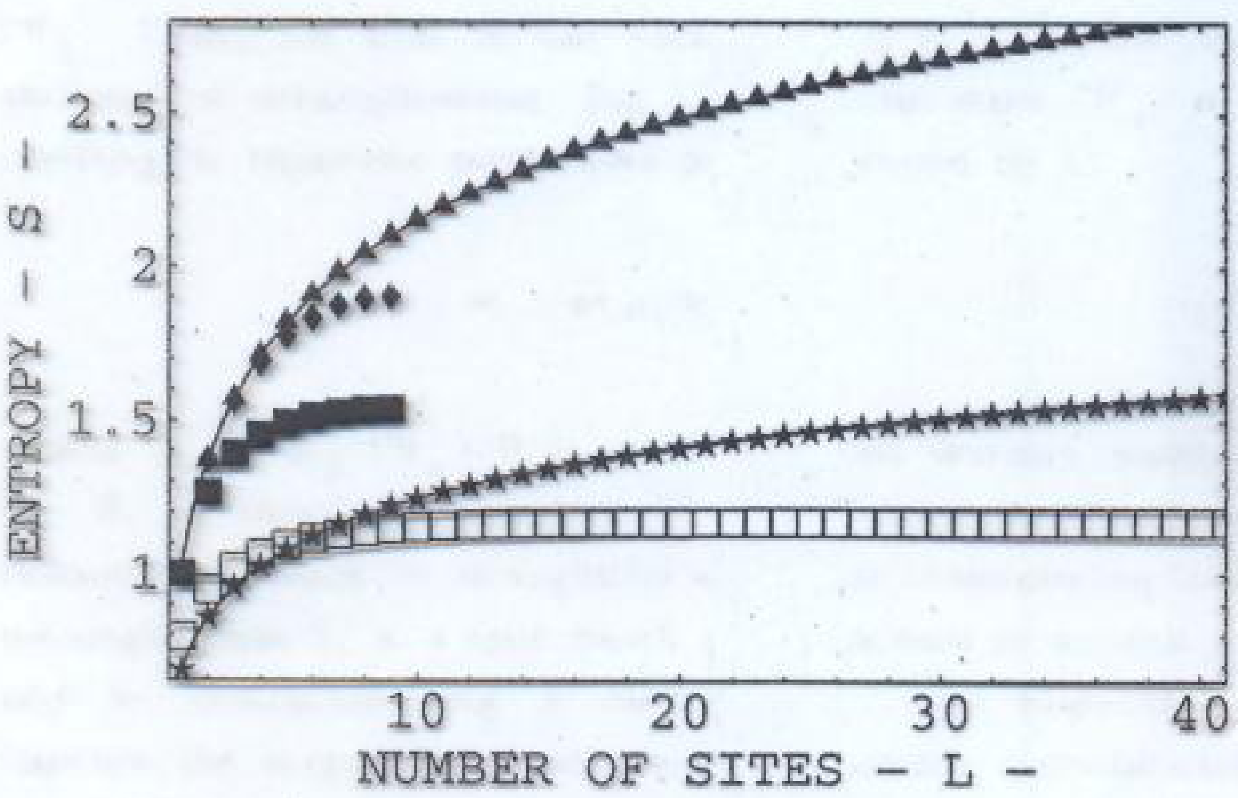


FIG. 1. *Noncritical entanglement* is characterized by a *saturation* of S_L as a function of the block size L: noncritical Ising chain (empty squares), $H_{XY}(a = 1.1, \gamma = 1)$; noncritical XXZ chain (filled squares), $H_{XXZ}(\Delta = 2.5, \lambda = 0)$. Instead, the entanglement of a block with a chain in a *critical* model displays a *logarithmic divergence* for large L: $S_L \sim \log_2(L)/6$ (stars) for the critical Ising chain, $H_{XY}(a = 1, \gamma = 1)$; $S_L \sim \log_2(L)/3$ (triangles) for the critical XX chain with no magnetic field, $H_{XY}(a = \infty, \gamma = 0)$; in a finite XXX chain of $N = 20$ spins without magnetic field (diamonds), $H_{XXZ}(\Delta = 1, \lambda = 0)$, S_L combines the critical logarithmic behavior for low L with a finite-chain saturation effect. We have also added the lines $[(c + \bar{c})/6][\log_2(L) + \pi]$ [cf. Eq. (12)] both for free conformal bosons (critical XX model) and free conformal fermions (critical Ising model) to highlight their remarkable agreement with the numerical diagonalization.

The XY model, Eq. (3), is an *exactly solvable* model (see for instance [13]) and this allows us to consider an infinite chain, $N \to \infty$. The calculation of S_L, as sketched next, also uses the fact that the ground state $|\Psi_g\rangle$ of the chain and the density matrices ρ_L for blocks of spins are *Gaussian* states that can be completely characterized by means of certain correlation matrix of second moments.

For each site l of the N-spin chain, we consider two Majorana operators, c_{2l} and c_{2l+1}, defined as

$$c_{2l} \equiv \left(\prod_{m=0}^{l-1} \sigma_m^z\right)\sigma_l^x; \qquad c_{2l+1} \equiv \left(\prod_{m=0}^{l-1} \sigma_m^z\right)\sigma_l^y. \tag{5}$$

Operators c_m are Hermitian and obey the anticommutation relations $\{c_m, c_n\} = 2\delta_{mn}$. Hamiltonian H_{XY} can be diagonalized by first rewriting it in terms of these new variables, $H_{XY}(\{\sigma_l^\alpha\}) \to H_{XY}(\{c_m\})$, and by then canonically transforming the operators c_m. The expectation value of c_m when the system is in the ground state, i.e., $\langle c_m\rangle \equiv \langle\Psi_g|c_m|\Psi_g\rangle$, vanishes for all m due to the Z_2 symmetry $(\sigma_l^x, \sigma_l^y, \sigma_l^z) \to (-\sigma_l^x, -\sigma_l^y, \sigma_l^z)\ \forall l$ of the original Hamiltonian H_{XY}. In turn, the expectation values $\langle c_m c_n\rangle = \delta_{mn} + i\Gamma_{mn}$ completely characterize $|\Psi_g\rangle$, for any other expectation value can be expressed, through Wick's theorem, in terms of $\langle c_m c_n\rangle$. Matrix Γ reads [16]

$$\Gamma = \begin{bmatrix} \Pi_0 & \Pi_1 & \cdots & \Pi_{N-1} \\ \Pi_{-1} & \Pi_0 & & \vdots \\ \vdots & & \ddots & \vdots \\ \Pi_{1-N} & \cdots & \cdots & \Pi_0 \end{bmatrix}, \qquad \Pi_l = \begin{bmatrix} 0 & g_l \\ -g_{-l} & 0 \end{bmatrix}, \tag{6}$$

with real coefficients g_l given, for an infinite chain $(N \to \infty)$, by

$$g_l = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{-il\phi}\, \frac{a\cos\phi - 1 - ia\gamma\sin\phi}{|a\cos\phi - 1 - ia\gamma\sin\phi|}. \tag{7}$$

From Eqs. (6) and (7) we can extract the entropy S_L of Eq. (4) as follows. First, from Γ, and by eliminating the rows and columns corresponding to those spins of the chain that do not belong to the block $\mathcal{B}_L$, we compute the correlation matrix of the state ρ_L, namely $\delta_{mn} + i(\Gamma_L)_{mn}$, where

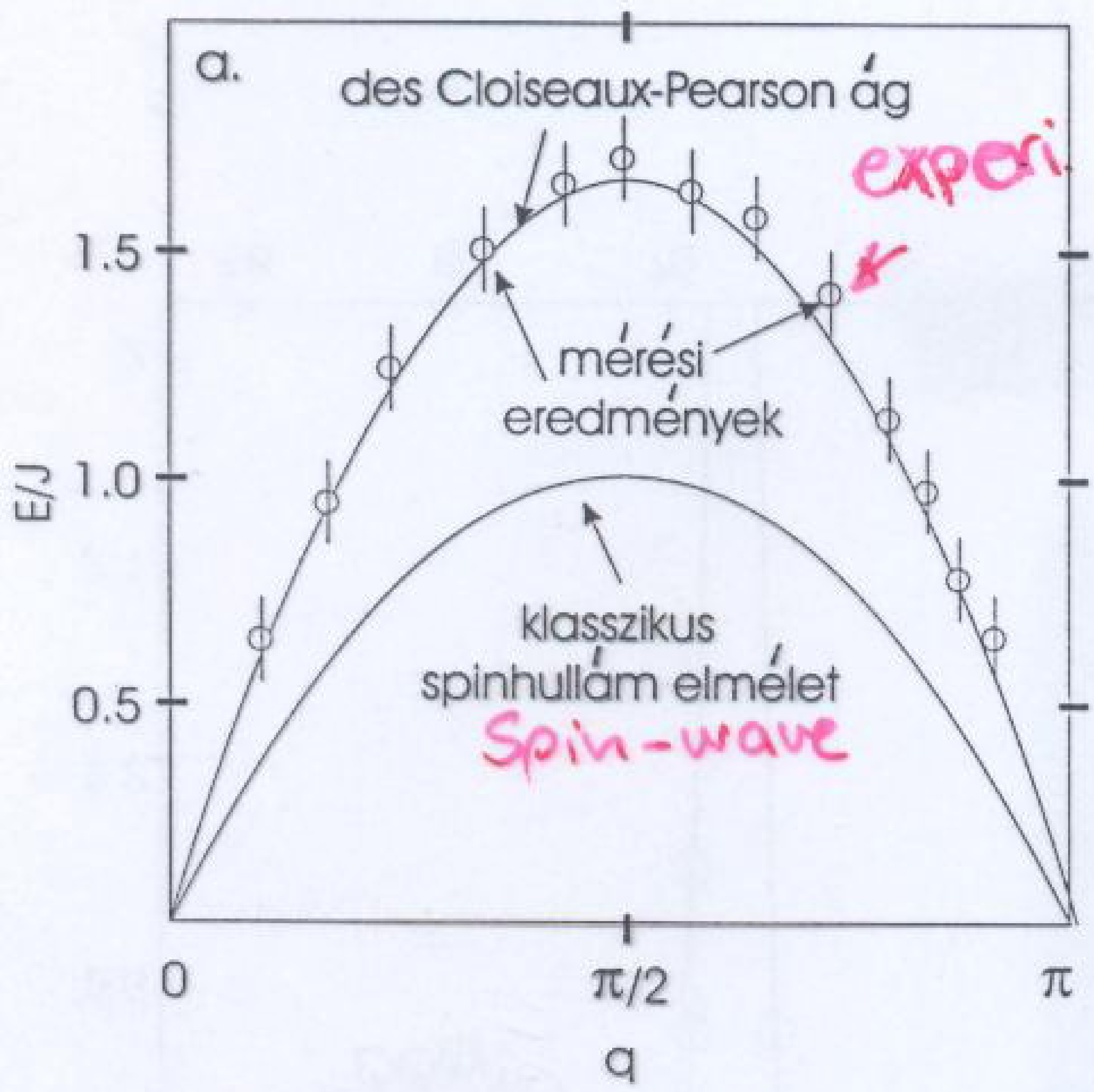
$$\Gamma_L = \begin{bmatrix} \Pi_0 & \Pi_1 & \cdots & \Pi_{L-1} \\ \Pi_{-1} & \Pi_0 & & \vdots \\ \vdots & & \ddots & \vdots \\ \Pi_{1-L} & \cdots & \cdots & \Pi_0 \end{bmatrix}. \tag{8}$$

Now, let $V \in \mathrm{SO}(2L)$ denote an orthogonal matrix that brings Γ_L into a block-diagonal form [19], that is

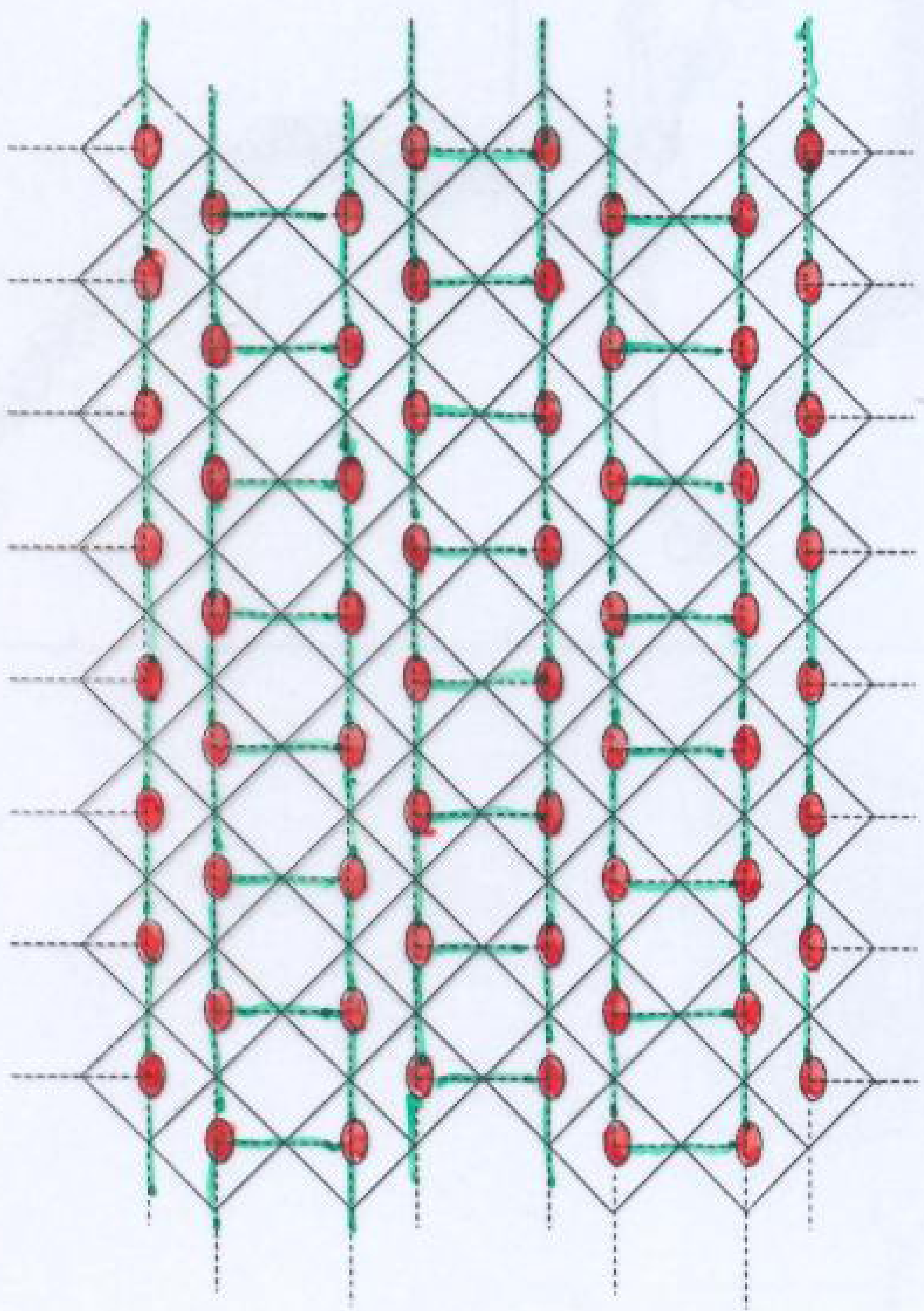
1-D

S-1/2

$CuCl_2 2N(C_5D_5)$



$\Delta = 0$



2 leg

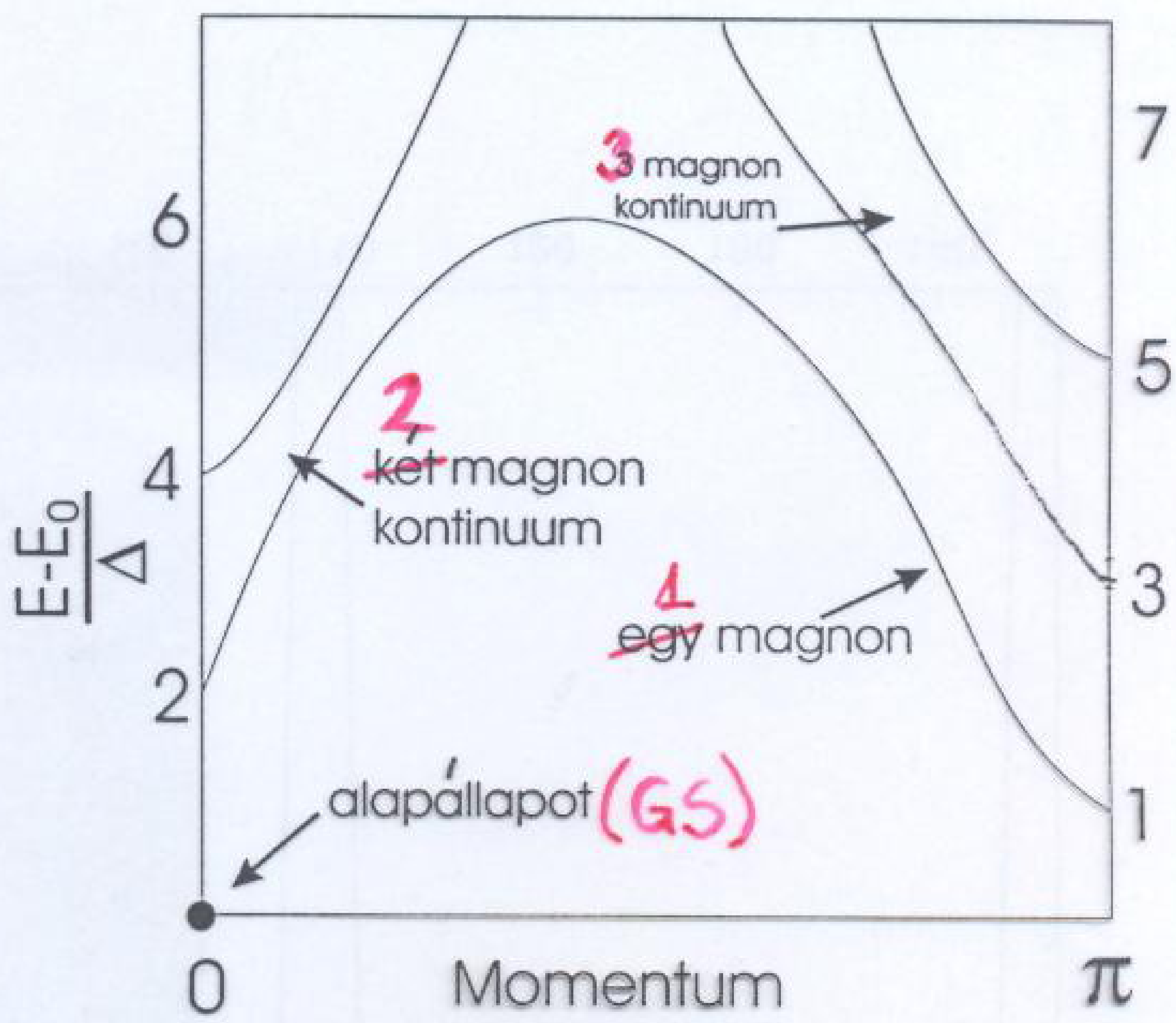
$(VO)_2P_2O_7$

$SrCu_2O_3$

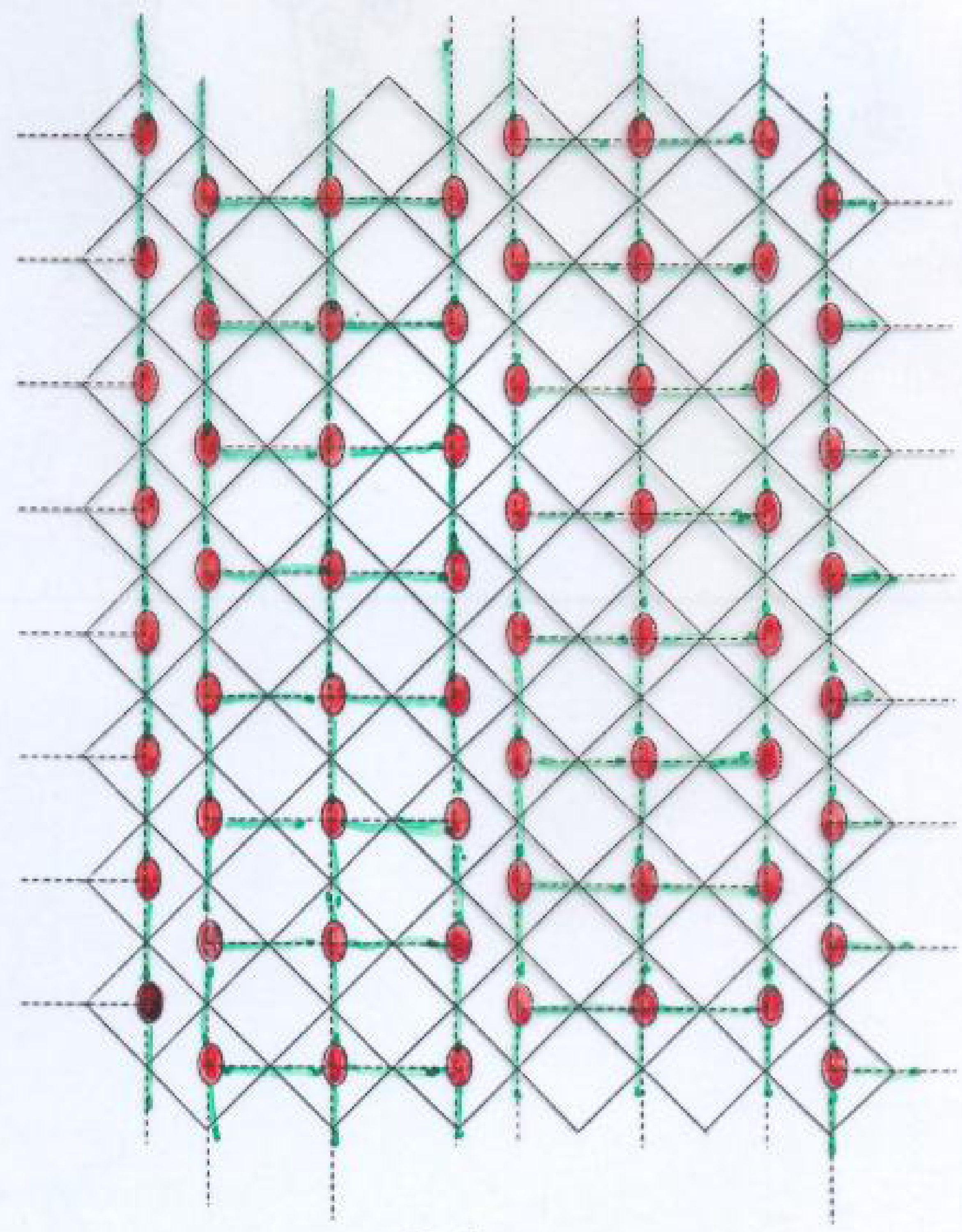
$\Delta \neq 0$

S-1

$CsNiCl_3$



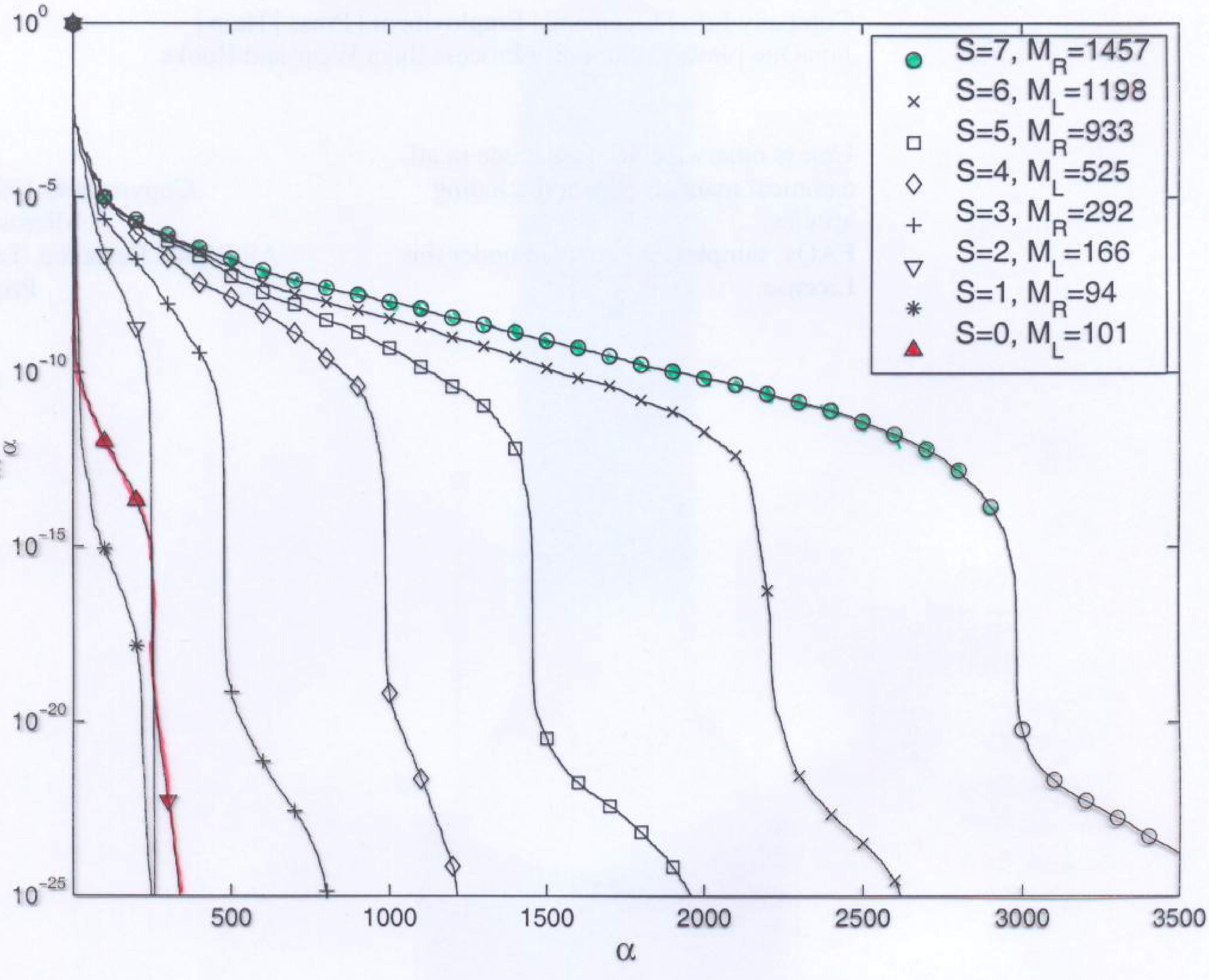
$\Delta_{Haldane}(k=\pi) = 0.4105$



3 leg

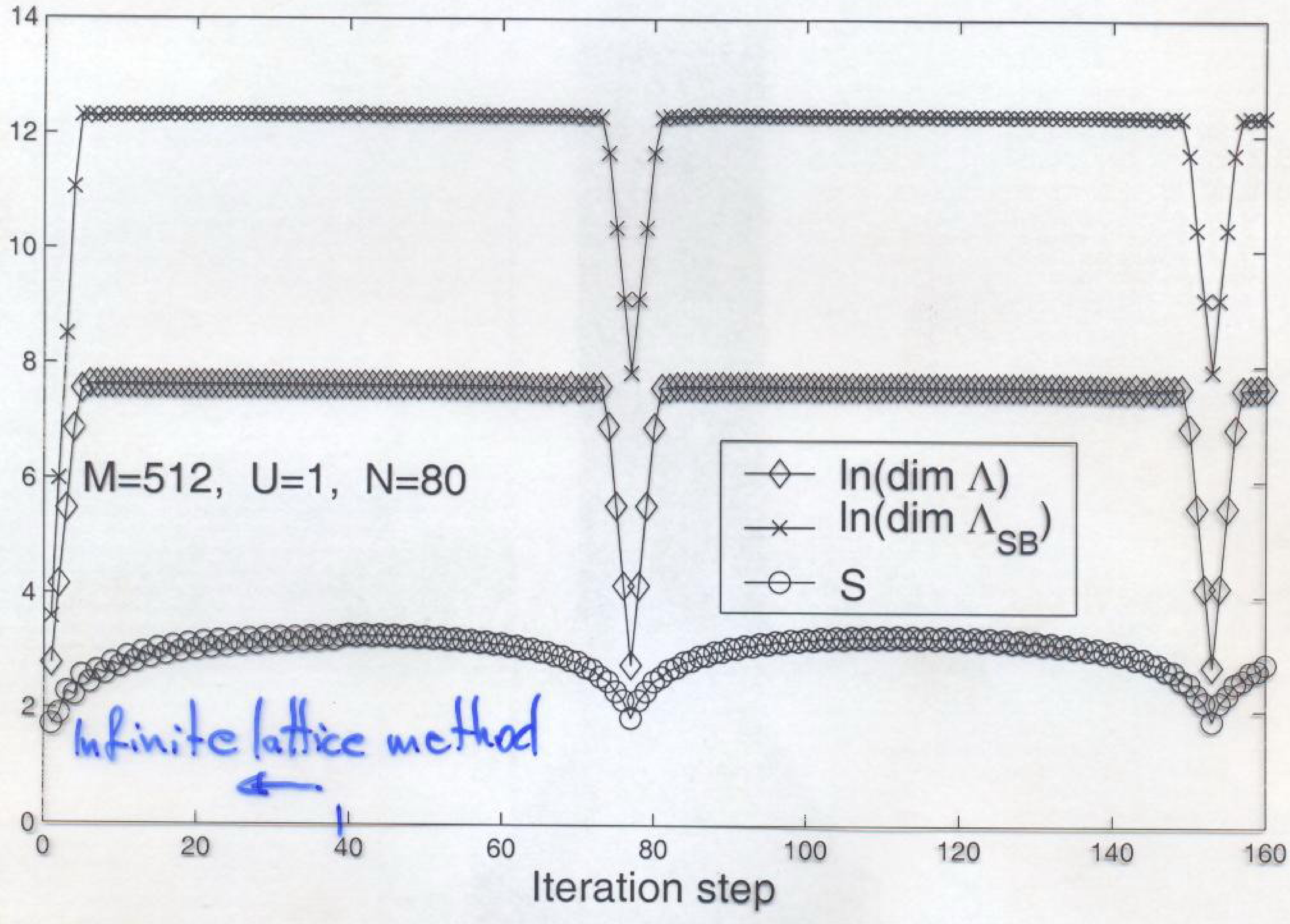
$SrCu_3O_5$

$\Delta = 0$

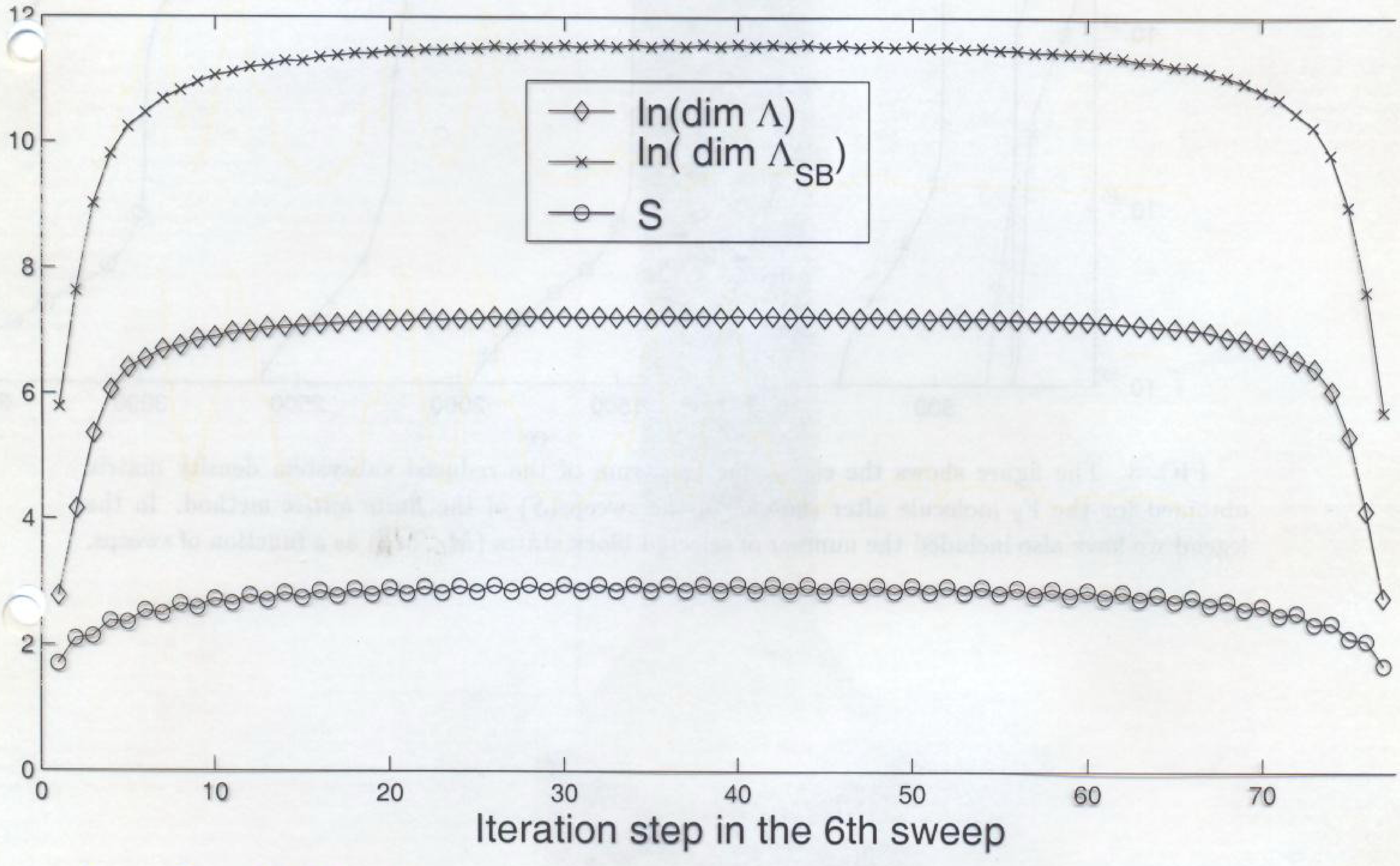


S=7, M_R=1457
S=6, M_L=1198
S=5, M_R=933
S=4, M_L=525
S=3, M_R=292
S=2, M_L=166
S=1, M_R=94
S=0, M_L=101
ω_α
α
10^{0}
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-25}
500
1000
1500
2000
2500
3000
3500

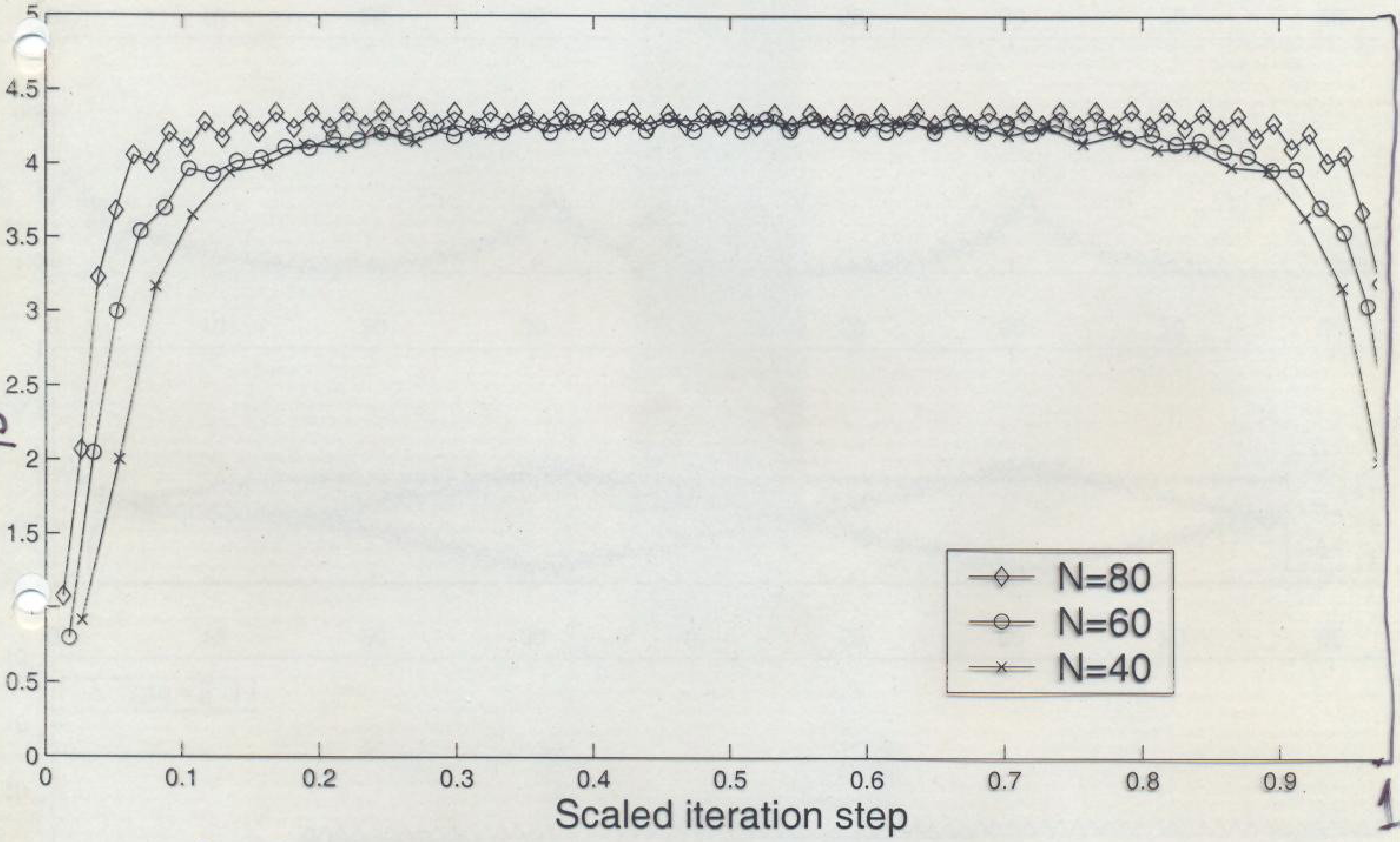
$$\mathcal{H} = \sum_{i\sigma} t\left(c^{+}_{i\sigma} c_{i+1\sigma} + c^{+}_{i+1\sigma} c_{i\sigma}\right) + U \sum_{i} c^{+}_{i\uparrow} c_{i\uparrow} c^{+}_{i\downarrow} c_{i\downarrow}$$



half-filled 1-d Hubbard model with PBC



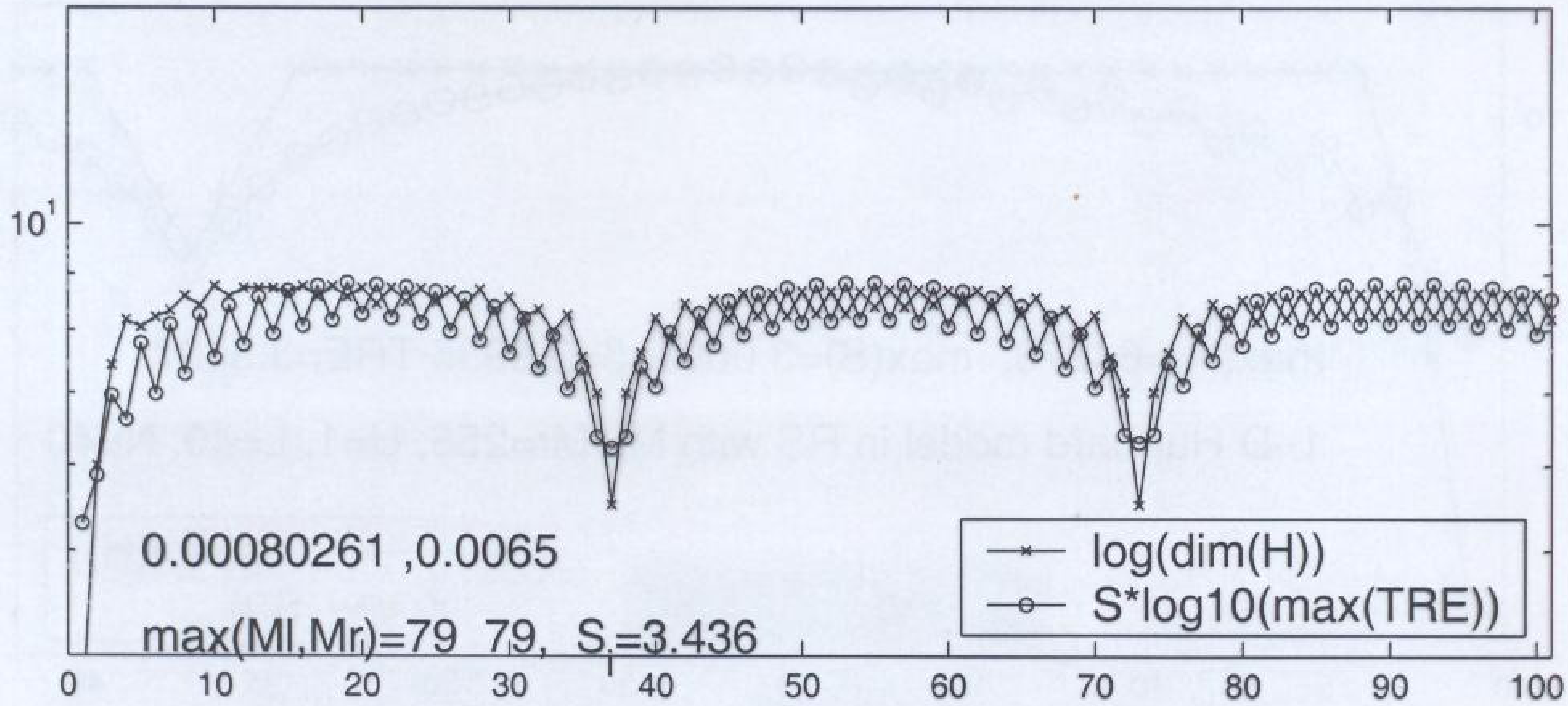
- half-filled 1-d Hubbard chain with PBC and DBSS



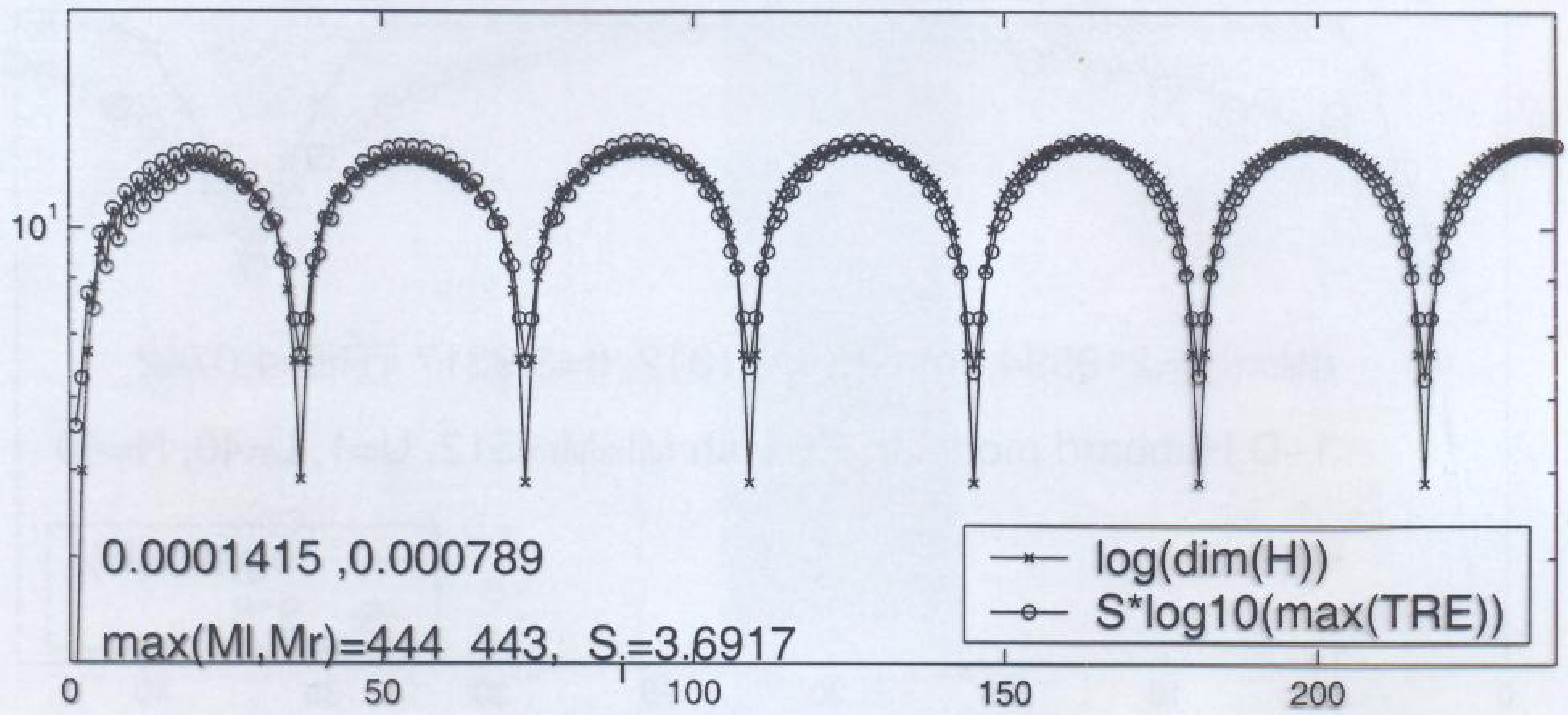
• half-filled Hubbard chain, DBSS

$TRE_{max} = 10^{-4}$, $M_{min} = 16$, $U = 1$

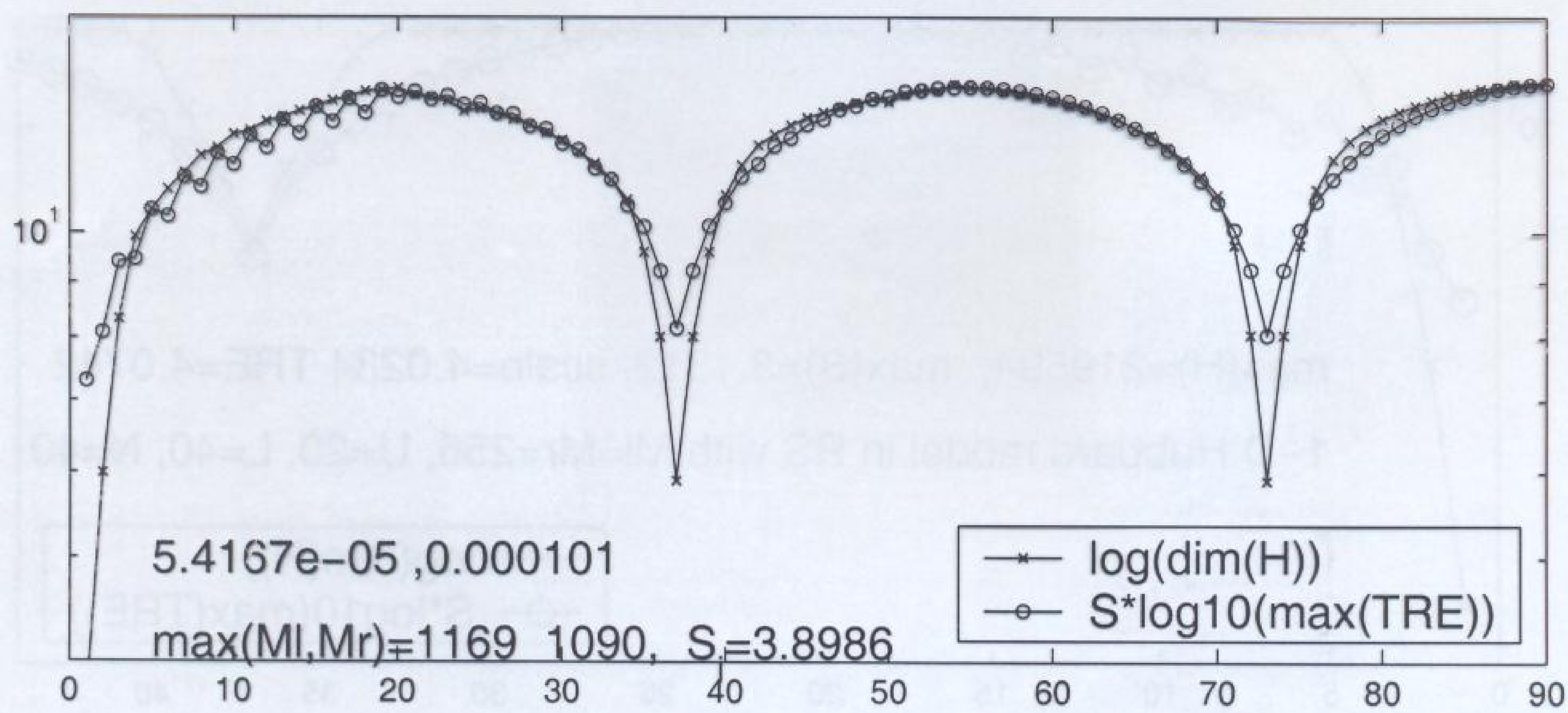
L=40, N=40, U=20, M_{min}=4



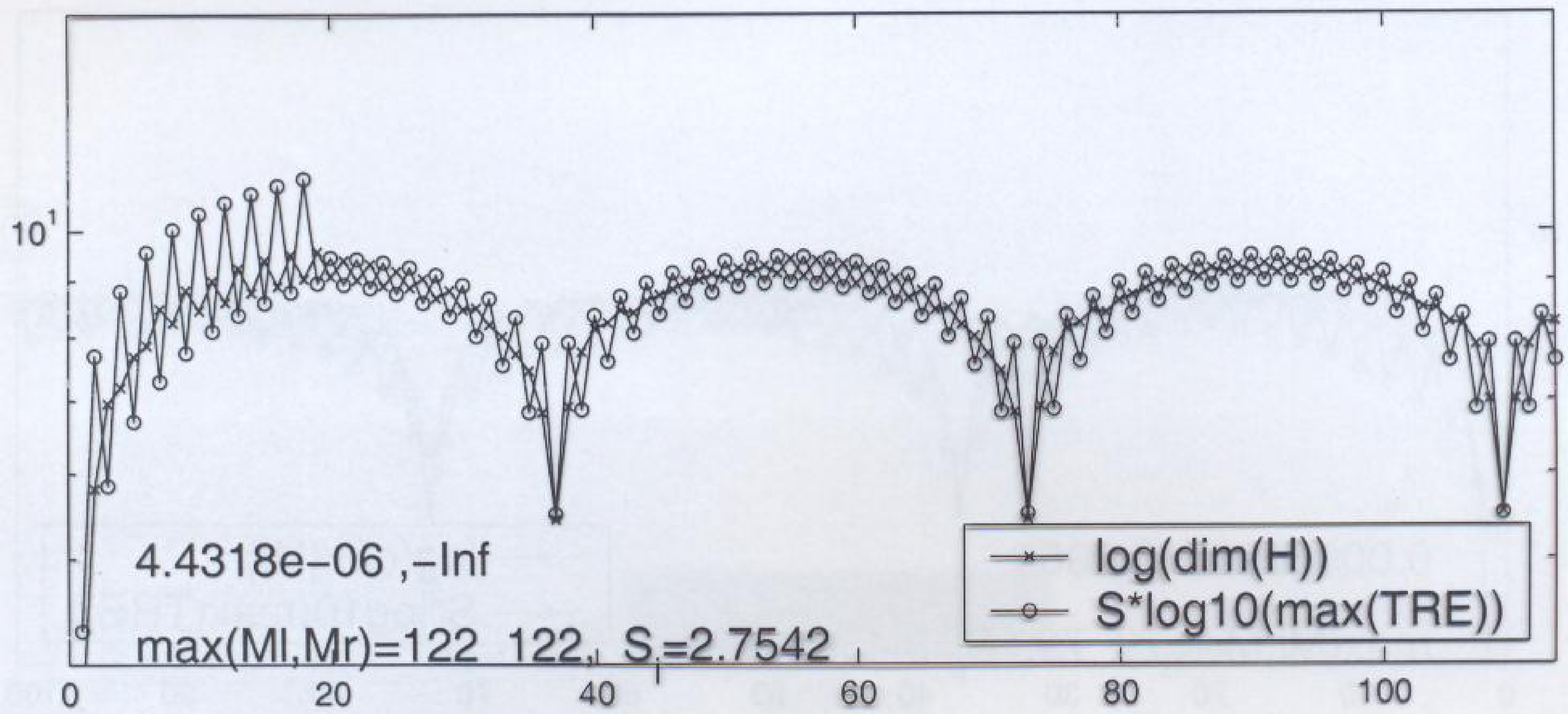
max(H)=6436, max(S)=2.8545, scale=3.0955 TRE=3.0022



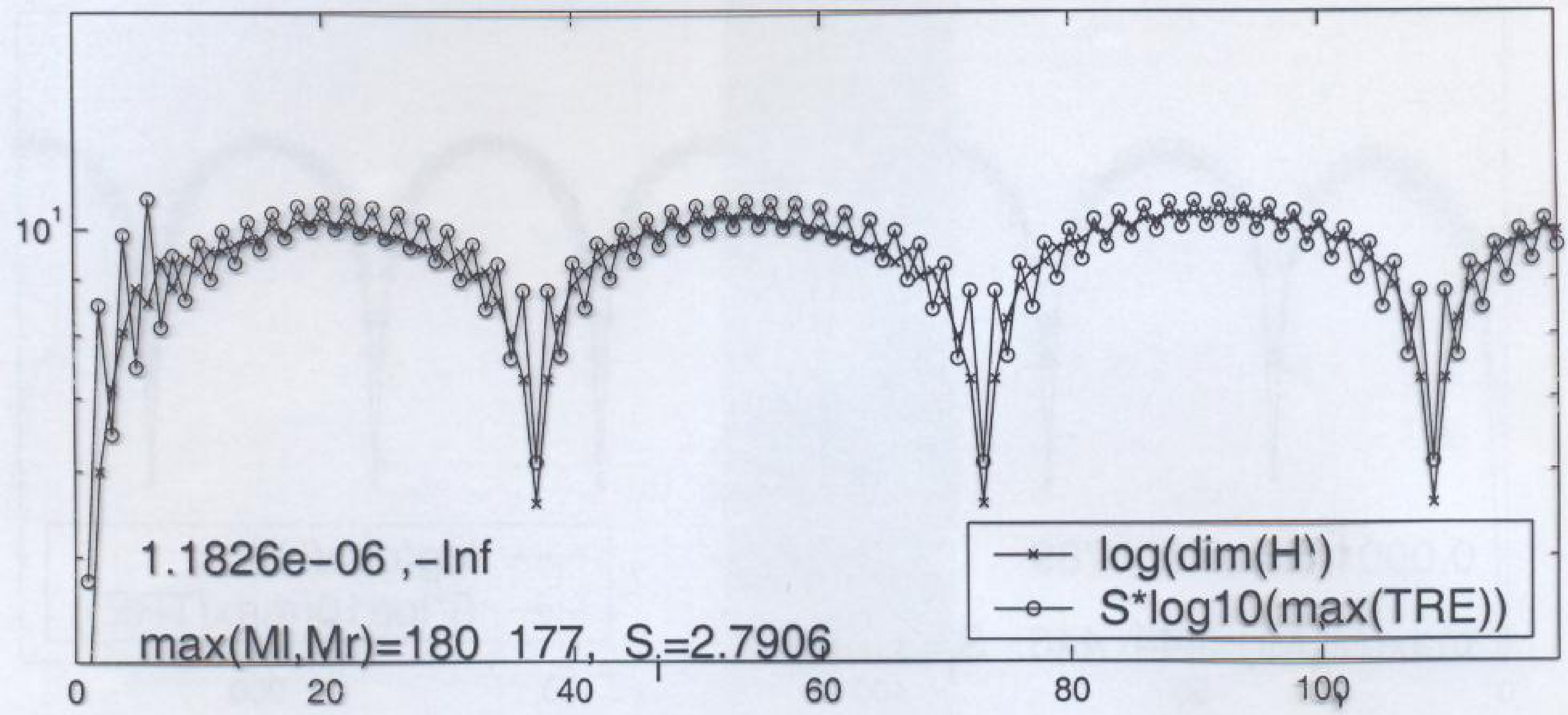
max(H)=170151, max(S)=3.1363, scale=3.8492 TRE=4.0026



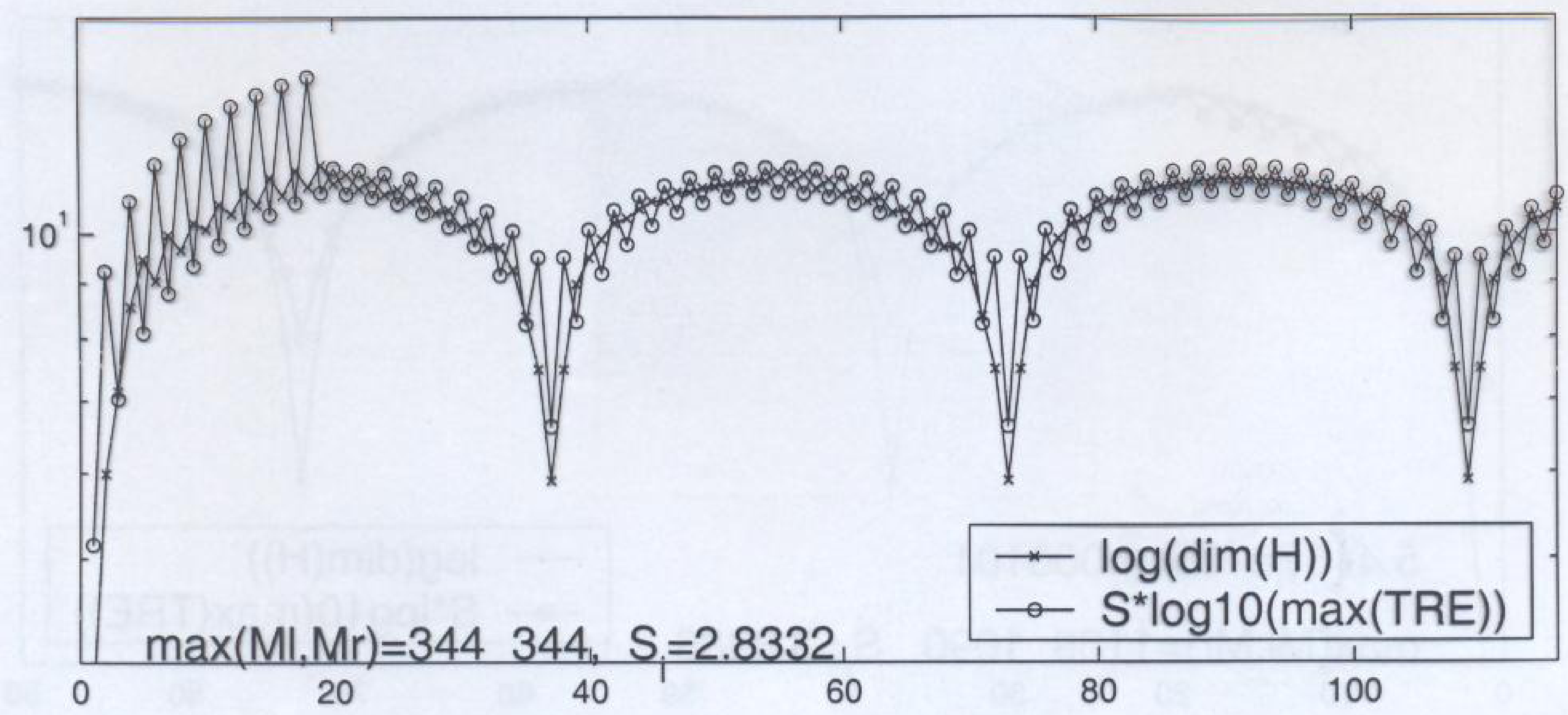
max(H)=991174, max(S)=3.2242, scale=4.2663 TRE=5



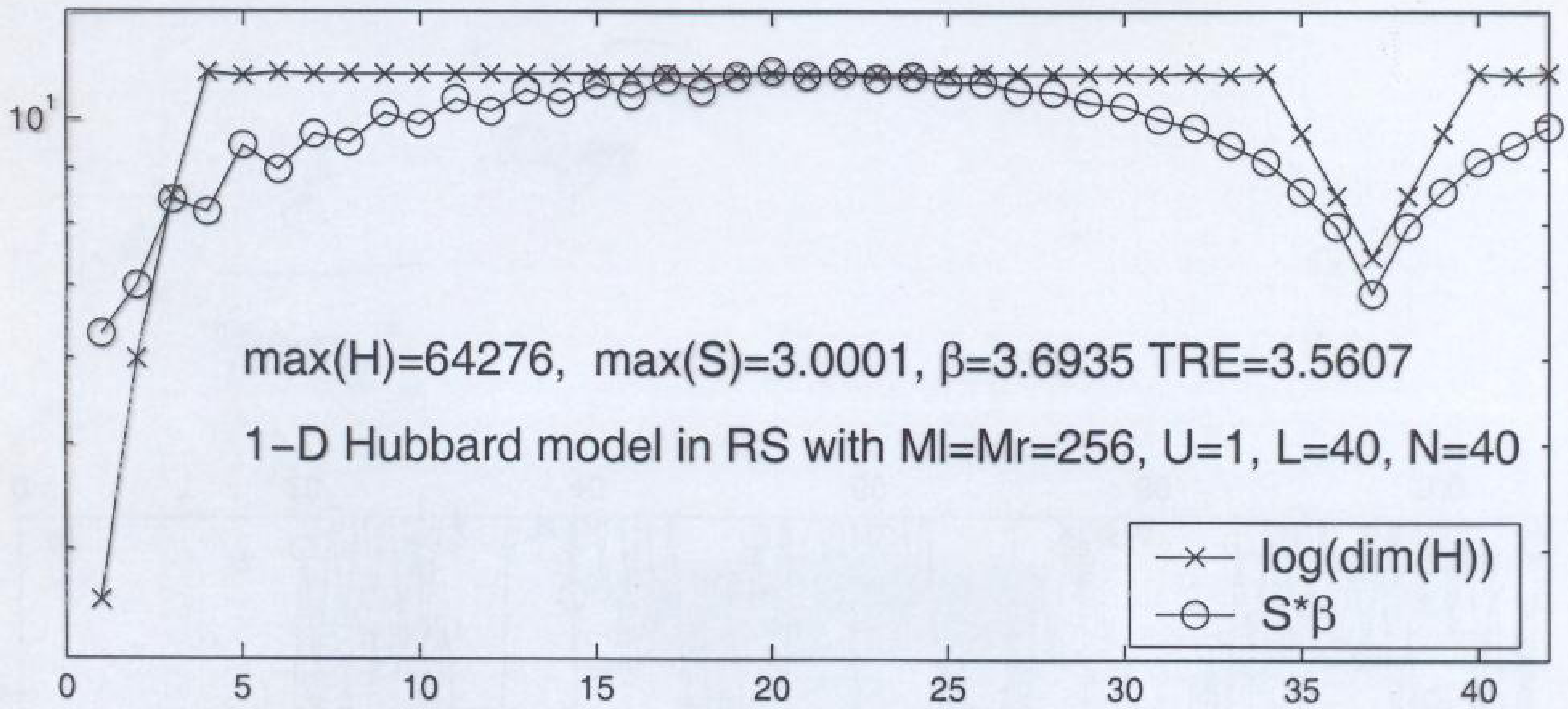
max(H)=14344, max(S)=2.0895, scale=5.3534 TRE=5

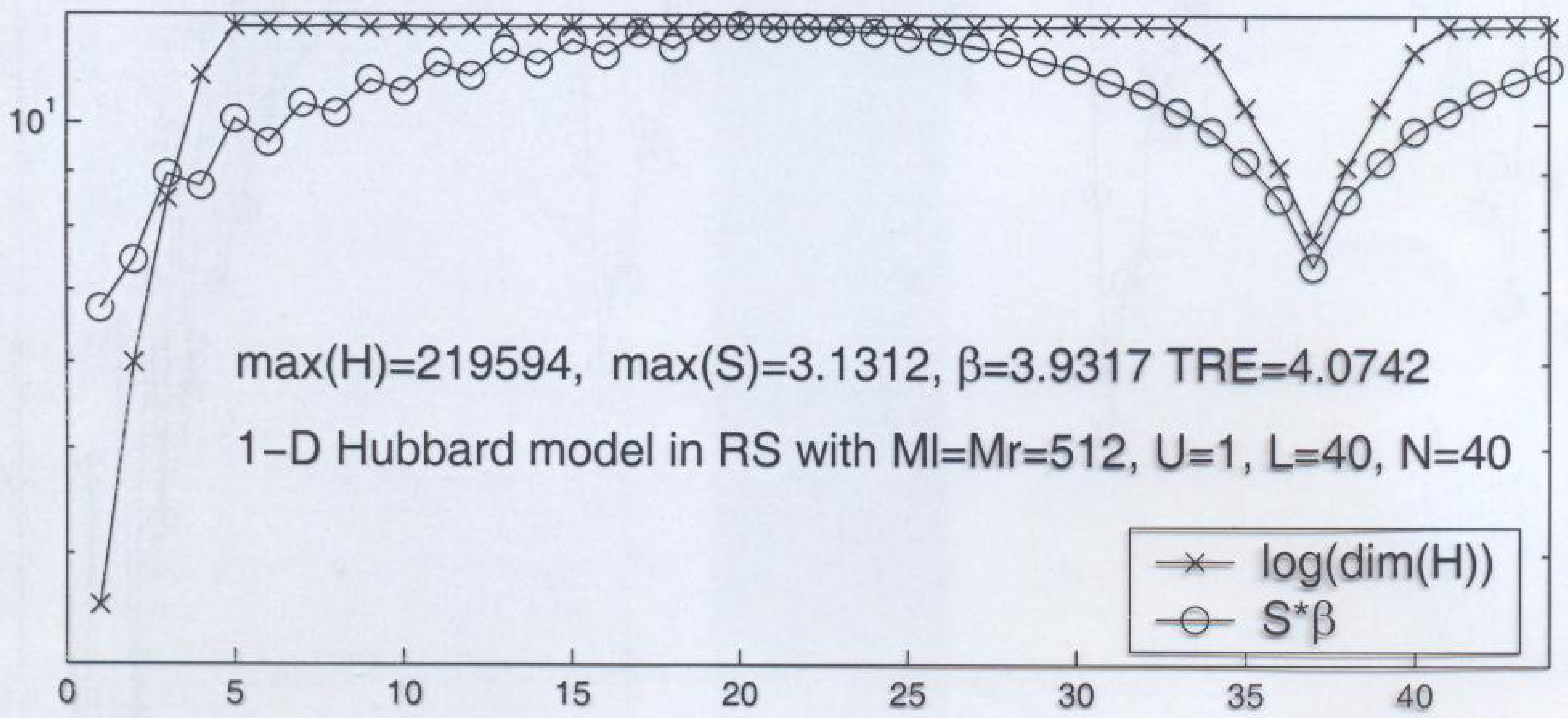


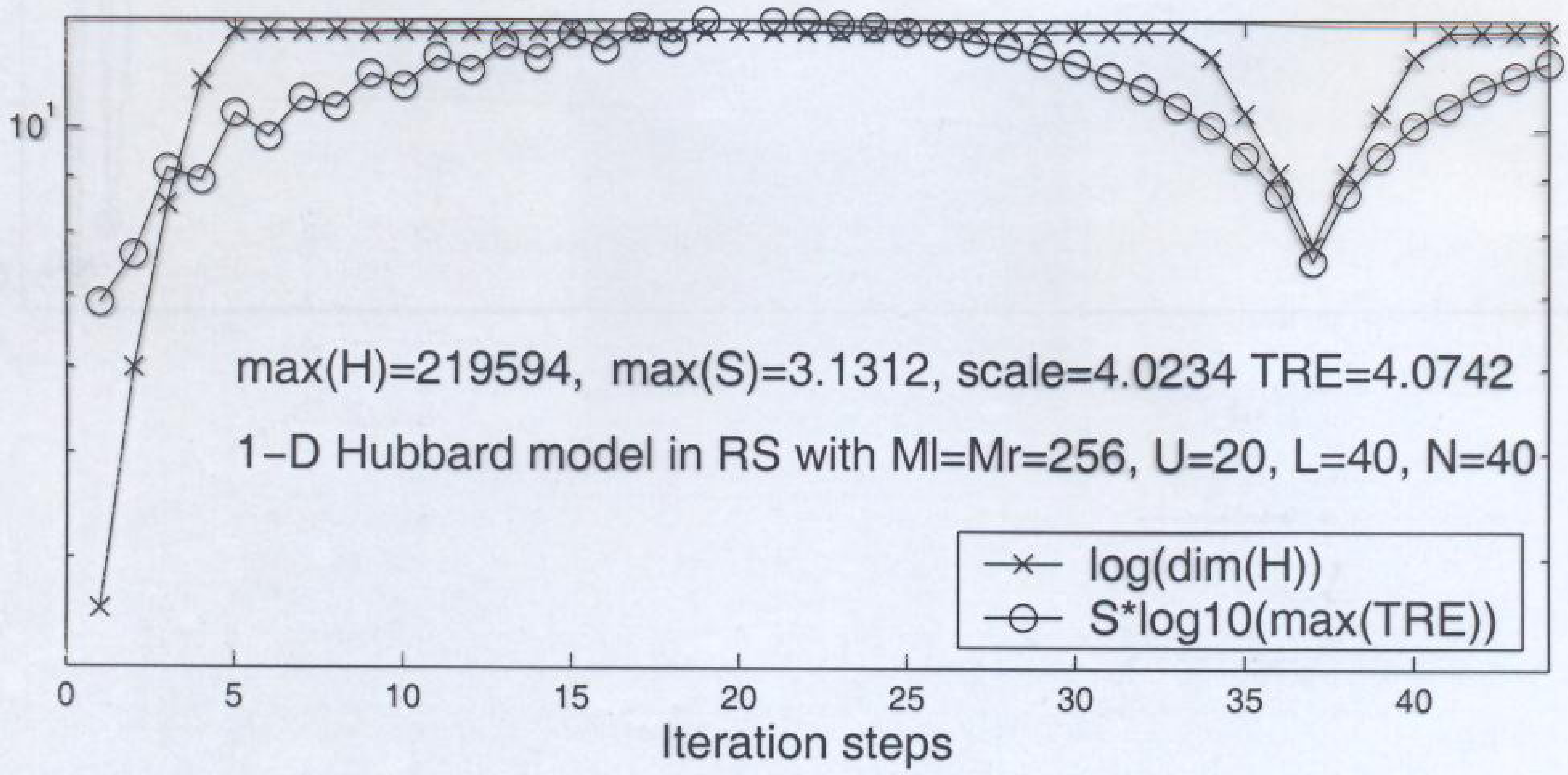
max(H)=30544, max(S)=1.7992, scale=5.9272 TRE=6.0004



max(H)=104276, max(S)=2.1713, scale=6.4234 TRE=7.0004


max(H)=64276, max(S)=3.0001, β=3.6935 TRE=3.5607
1-D Hubbard model in RS with Ml=Mr=256, U=1, L=40, N=40
log(dim(H))
S*β
10^1
0
5
10
15
20
25
30
35
40


max(H)=219594, max(S)=3.1312, β=3.9317 TRE=4.0742
1-D Hubbard model in RS with Ml=Mr=512, U=1, L=40, N=40
log(dim(H))
S*β
10^1
0
5
10
15
20
25
30
35
40


max(H)=219594, max(S)=3.1312, scale=4.0234 TRE=4.0742
1-D Hubbard model in RS with Ml=Mr=256, U=20, L=40, N=40
log(dim(H))
S*log10(max(TRE))
10^1
0
5
10
15
20
25
30
35
40
Iteration steps